FACULTY OF SCIENCE
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# Masters Thesis in Mathematics-Economics

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## Costly access to credit markets in a DSGE model with involuntary unemployment

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# Abstract

During the great recession starting in 2008, governments were often encouraged to apply pressure onto banks to make them extend more credit. The reasoning was that greater access to credit would allow households and businesses to consume and invest, thus allowing the economy to recover quicker. In this thesis, a game theoretical model of households is embedded within a DSGE model to investigate this mechanism of government pressure on banks.

We assume differentiated households, each distinguished from the other by their job- and credit aversion. Households participate in a job market game and after this participate in a credit market game. A family is formed after the results of the games are revealed, and the family distributes consumption among employed and unemployed. The family consists of two segments, based on household outcomes. One section is rule of thumb, the other is optimising.

The family sets employment levels and consumption levels according to incentive compatibility to optimise its utility function. The family owns the producers in the economy, which are subject to Calvo price frictions. There is no government expenditure in the model.

The model proves difficult to solve explicitly, but preliminary analysis indicates several mechanisms by which government pressure on banks can affect aggregate output. Additionally, several complicated interaction patterns arise between the rule of thumb segment and the optimising segment not seen in other models. Finally, the family's optimisation problem turns out to exhibit state dependency, despite no capital being assumed in the model.

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# Chapter 1

# Introduction

One of the persistent stories in the news during the Great Recession after the Financial Crisis was small businesses and households unable to obtain credit. Owners of businesses complained that banks were calling in loans, and refusing credit for new expansion, households were unable to obtain mortgages. Government intervention was often suggested as a possible remedy, either through the large government ownership stakes in banks, or through regulation, to make banks extend credit to a greater extent. In this thesis, a model will be constructed to examine the linkages between credit availability, unemployment, and economic activity, and answer the primary question whether government intervention to increase credit availability can help an economy recover from a downturn.

The theoretical basis for this model is to be found primarily in two different papers. From [Galí et al., 2007] we use the concept of rule of thumb households, who are unable to obtain credit and thus must consume their total income every period. Meanwhile, from [Christiano et al., 2010], we expand on a game-theoretical model of involuntary unemployment, where households must invest effort into obtaining a job, and unemployed households receive support from the employed households in a family construct.

The structure of the thesis is as follows:

In chapter 2, a model of household behaviour is developed. Each household is different from the next, differentiated in two parameters. Households individually seek to maximise expected utility, and six distinct strategies emerge based on this principle.

In chapter 3, the family is introduced, which anticipates and predicts household behaviour. The family enforces redistribution from employed to unemployed, but acts in the aggregate best interests of households.

In chapter 4, a reduced model is developed, where, by removing the possibility of obtaining credit when unemployed, restrict the available strategies from six to four. This simplifies the model significantly, and brings with it a dimensional reduction of a matrix that would otherwise have been even more problematic than it turns out to be.

In chapter 5, we flesh out the production sector of the DSGE model by proposing the simplest possible setup: A continuum of local monopolists produce intermediate goods and are subject to Calvo price frictions. The standard structure of this half of the model allows us to keep our focus on the family through the rest of the model.

In chapter 6, we consider the equilibrium of the DSGE model, and by combining clearing in the various markets with the optimality conditions, we get a DSGE system that seems to reduce to two real variables: inflation and size of the optimising family. Unfortunately, several surprises pop up along the way, preventing us from obtaining a closed form solution to the model.

In chapter 7, we consider the behaviour of consumption and employment at natural levels and in steady state, and in chapter 8 we propose an initial calibration of the model.

Finally in chapter 9, we use the model to examine the impact of a change in government pressure on banks. We find preliminary evidence in support of the hypothesis that government pressure on banks may be warranted, and summarise, conclude and comment on limitations in chapter 10.

Additionally, four appendices are included, documenting the calculations and programming done with regards to bridging the gap between the game theoretical model, and the macroeconomic model.

# Chapter 2

# A Model of the Households

## 2.1 Introduction

The primary focus of this DSGE model is on the interaction between participation in the job market and the credit market among households. This chapter develops a two-stage game where households participate first in the job market game, and then in the credit market game. After these games have been played out, a family is formed consisting of all households. The households can only interact with the rest of the model through the family.

The game developed below is an extension of the game developed in [Christiano et al., 2010], where a unit length of households participate in the job market game, and are possibly subject to involuntary unemployment. Our model keeps the possibility of involuntary unemployment, and additionally adds involuntary exclusion from the credit market.

## 2.2 A model of the households

Assume a continuum of households. Each period, a household draws two independent efficiency parameters $\phi, \psi$ from the uniform distribution over $[0, 1]$. Consequently we may think of the households as populating the unit square. The parameters are respectively aversion to work and cost of credit maintenance. If the household is employed, it thus must sustain a disutility of $F^j + \varsigma_j(1 + \sigma_j)\phi^{\sigma_j}$. In addition, if a household has obtained credit, it must sustain a disutility of $F^k + \varsigma_k(1 + \sigma_k)\psi^{\sigma_k}$. We assume $F_j, F_k \in \mathbb{R}^+$, and $\sigma_j, \sigma_k, \varsigma_j, \varsigma_k \in \mathbb{R}^+$.

Each household, after drawing its privately observed efficiency parameters, must decide whether to participate in the job market or not. Thus the household chooses an effort level $e_t^j \in [0, \infty[$. The higher the effort level, the higher the probability of finding a job:[1]

$$p(e_t^j) = \eta + a_e e_t^j.$$

Effort is costly, meaning the household incurs a quadratic disutility of effort given by $\frac{1}{2}(e_t^j)^2$.

After the results of the job market game are revealed (each household achieves $j_t \in \{0, 1\}$), households must decide whether to participate in the credit market or not. Again, households choose an effort level $e_t^k \in [0, \infty[$, which influences the probability $q$ of obtaining credit.

The probability $q$ depends upon employment status (banks will be more willing to extend credit to employed households), the effort parameter chosen, a policy parameter (to model government pressure upon banks to lend more), and the output gap (which functions as a crude proxy of banks' fear of bankruptcy among clients). Thus we have:[2]

$$q(j_t, e_t^k) = b_j j_t + b_g g_t(j_t) + b_e e_t^k - b_x x_{t-1}.$$

Finally, a single family is formed consisting of all households irrespective of outcome. As [Christiano et al., 2010] notes: "We view the family as a stand-in for various market and non-market arrangements that actual households have for dealing with idiosyncratic labor market outcomes." The family cannot observe the effort choices made by households, only the outcomes obtained.

Based on outcome, the family must redistribute between employed and unemployed households by setting post-redistribution consumption levels enjoyed by the households as

$$c_t(j_t, k_t).$$

We assume though that redistribution can not happen from optimising ($k_t = 1$) to rule of thumb households ($k_t = 0$) and vice versa, this ensures that rule of thumb households truly remain rule of thumb, and cannot simply access the credit market by proxy through the optimising segment of the family.

## 2.3 Private outcomes for the households

Based on the model developed above, we can list out the possible final outcomes for a household described by its two efficiency parameters $\phi$ and $\psi$.

[1] We note already that calibration of the model must ensure no household obtains $p(e_t^j) > 1$.

[2] We likewise need to ensure that our chosen calibration leaves this probability between zero and one.

Listing out the outcomes allows us in the next sections to partition households according to their chosen strategies.

### 2.3.1 The rule of thumb outcomes

Common among rule of thumb households is that they will not be able to borrow or save. Several possible outcomes are present within the class of rule of thumb households, each with possibly different ex-post utility:

2. Succeeded in $j$ game. Participated in $k$ game, failed.
   Utility: $U_2 = \log\left(c_t(1,0)\right) - F^j - \varsigma_j(1+\sigma_j)\phi^{\sigma_j} - \frac{1}{2}(e_t^j)^2 - \frac{1}{2}(e_t^k)^2$
3. Succeeded in $j$ game. Did not participate in $k$ game.
   Utility: $U_3 = \log\left(c_t(1,0)\right) - F^j - \varsigma_j(1+\sigma_j)\phi^{\sigma_j} - \frac{1}{2}(e_t^j)^2$
5. Participated in $j$ game, failed. Participated in $k$ game, failed.
   Utility: $U_5 = \log\left(c_t(0,0)\right) - \frac{1}{2}(e_t^j)^2 - \frac{1}{2}(e_t^k)^2$
6. Participated in $j$ game, failed. Did not participate in $k$ game.
   Utility: $U_6 = \log\left(c_t(0,0)\right) - \frac{1}{2}(e_t^j)^2$
8. Did not participate in $j$ game. Participated in $k$ game, failed.
   Utility: $U_8 = \log\left(c_t(0,0)\right) - \frac{1}{2}(e_t^k)^2$
9. Did not participate in either game.
   Utility: $U_9 = \log\left(c_t(0,0)\right)$

Of these, only the first two types are employed, and the family will make these types support the other types. Note that we have ignored possible mixed strategies for the households.

### 2.3.2 The optimising outcomes

Only three different outcomes allow the household to be optimising:

1. Succeeded in $j$ game. Succeeded in $k$ game.
   Utility: $U_1 = \log\left(c_t(1,1)\right) - F^j - \varsigma_j(1+\sigma_j)\phi^{\sigma_j} - F^k - \varsigma_k(1+\sigma_k)\psi^{\sigma_k} - \frac{1}{2}(e_t^j)^2 - \frac{1}{2}(e_t^k)^2$
4. Participated in $j$ game, failed. Succeeded in $k$ game.
   Utility: $U_4 = \log\left(c_t(0,1)\right) - F^k - \varsigma_k(1+\sigma_k)\psi^{\sigma_k} - \frac{1}{2}(e_t^j)^2 - \frac{1}{2}(e_t^k)^2$

7. Did not participate in $j$ game. Succeeded in $k$ game.
   Utility: $U_7 = \log\left(c_t(0,1)\right) - F^k - \varsigma_k(1+\sigma_k)\psi^{\sigma_k} - \frac{1}{2}(e_t^k)^2$

Among these types, the first two are employed, and the family will make these types support the third type.

## 2.4 The household's maximisation problem

In line with standard assumptions, households will maximise their expected utility. To solve this maximisation problem, we proceed by backwards induction. Thus we begin with the credit market game.

### 2.4.1 The credit market game

#### Solving the maximisation problem

A household which has obtained a job ($j_t = 1$) and chooses to participate in the credit market game to obtain credit ($k_t = 1$) solves the following problem of maximising expected utility:

$$
\begin{aligned}
&\max_{e_t^k}\left[q(1,e_t^k)U_1 + (1-q(1,e_t^k))U_2\right] \\
=&\max_{e_t^k}\left[q(1,e_t^k)(U_1-U_2)+U_2\right] \\
=&\max_{e_t^k}\left[q(1,e_t^k)\left(\log\left(\frac{c_t(1,1)}{c_t(1,0)}\right) - F^k - \varsigma_k(1+\sigma_k)\psi^{\sigma_k}\right) - \frac{1}{2}(e_t^k)^2\right],
\end{aligned}
$$

where we have removed irrelevant constants along the way, since these are independent of effort.

Thus we obtain an optimal effort level for employed households:

$$
\begin{aligned}
e_t^{k,1} &= \max\left[b_e\left(\log\left(\frac{c_t(1,1)}{c_t(1,0)}\right) - F^k - \varsigma_k(1+\sigma_k)\psi^{\sigma_k}\right), 0\right] \\
&= \max\left[b_e\varsigma_k(1+\sigma_k)\left((m_t^{k,1})^{\sigma_k} - \psi^{\sigma_k}\right), 0\right].
\end{aligned}
$$

Where $m_t^{k,1}$ will be explained in the next subsection.

Similarly for unemployed households and households outside the labour force, we will obtain:

$$
\begin{aligned}
e_t^{k,0} &= \max\left[b_e\left(\log\left(\frac{c_t(0,1)}{c_t(0,0)}\right) - F^k - \varsigma_k(1+\sigma_k)\psi^{\sigma_k}\right), 0\right] \\
&= \max\left[b_e\varsigma_k(1+\sigma_k)\left((m_t^{k,0})^{\sigma_k} - \psi^{\sigma_k}\right), 0\right].
\end{aligned}
$$

Where $m_t^{k,0}$ is also explained in the next subsection.

Note that expected utility from participation, and thus optimal effort levels in the game are dependent on $\phi$, but independent of $\psi$.

### The marginal households in the credit market game

Certain households will choose not to participate in the credit market game if the expected return from participation is not high enough. By the principle of expected utility maximisation, households with an increase in ex-ante utility from participation will participate. The rest will stay outside. Due to this, there exists two marginal households with $\psi = m_t^{k,j_t}$ respectively characterised by equality of expected utilities from participating or not:

$$\begin{aligned}
0 =& q\left(1, e_t^{k,1}\right) U_1 + \left(1 - q\left(1, e_t^{k,1}\right)\right) U_2 - U_3 \\
=& q\left(1, e_t^{k,1}\right) (U_1 - U_2) + (U_2 - U_3) \\
=& q\left(1, e_t^{k,1}\right) \left(\log\left(\frac{c_t(1,1)}{c_t(1,0)}\right) - F^k - \varsigma_k(1+\sigma_k)(m_t^{k,1})^{\sigma_k}\right) - \frac{1}{2}(e_t^{k,1})^2 \\
=& \left(b_j + b_g g_t(1) - b_x x_{t-1}\right)\left(\log\left(\frac{c_t(1,1)}{c_t(1,0)}\right) - F^k - \varsigma_k(1+\sigma_k)(m_t^{k,1})^{\sigma_k}\right) \\
&+ \frac{1}{2}\left(\max\left[b_e\left(\log\left(\frac{c_t(1,1)}{c_t(1,0)}\right) - F^k - \varsigma_k(1+\sigma_k)(m_t^{k,1})^{\sigma_k}\right), 0\right]\right)^2
\end{aligned}$$

To solve the equation above, we note that $(b_j + b_g g_t(1) - b_x x_{t-1})$ is non-zero in general, and since the second term is non-negative, we are left with the conclusion that the marginal household can be described by:

$$\log\left(\frac{c_t(1,1)}{c_t(1,0)}\right) = F^k + \varsigma_k(1+\sigma_k)(m_t^{k,1})^{\sigma_k}.$$

Likewise we obtain, from repeating the calculations for households that are not employed:

$$\log\left(\frac{c_t(0,1)}{c_t(0,0)}\right) = F^k + \varsigma_k(1+\sigma_k)(m_t^{k,0})^{\sigma_k}.$$

### Ex-ante utilities

It will be useful later to consider the ex-ante utilities of households as they make their choices in the credit market game.

Employed households with credit aversion $\psi \leq m_t^{k,1}$ will thus have the following ex-ante utility:

$$
\begin{aligned}
U_{A1} =& q\left(1, e_t^{k,1}\right) U_1 + \left(1 - q\left(1, e_t^{k,1}\right)\right) U_2 \\
=& q\left(1, e_t^{k,1}\right)(U_1 - U_2) + U_2 \\
=& q(1, e_t^{k,1})\left(\log\left(\frac{c_t(1,1)}{c_t(1,0)}\right) - F^k - \varsigma_k(1+\sigma_k)\psi^{\sigma_k}\right) \\
& + \log c_t(1,0) - F^j - \varsigma_j(1+\sigma_j)\phi^{\sigma_j} - \frac{1}{2}(e_t^{k,1})^2 - \frac{1}{2}(e_t^j)^2 \\
=& k_1 f_k\left(m_t^{k,1}, \psi\right) + \frac{1}{2} b_e^2 f_k^2\left(m_t^{k,1}, \psi\right) \\
& + \log c_t(1,0) - F^j - \varsigma_j(1+\sigma_j)\phi^{\sigma_j} - \frac{1}{2}(e_t^j)^2
\end{aligned}
$$

where we have introduced the notation $k_1 = (b_j + b_g g_t(1) - b_x x_{t-1})$ along with $f_k(m, \psi) = \varsigma_k(1+\sigma_k)\left(m^{\sigma_k} - \psi^{\sigma_k}\right)$, which will be used extensively.

Similarly, unemployed households with credit aversion $\psi \leq m_t^{k,0}$ will have, with $k_0 = (b_g g_t(0) - b_x x_{t-1})$:

$$
\begin{aligned}
U_{A0} =& q\left(0, e_t^{k,0}\right) U_4 + \left(1 - q\left(1, e_t^{k,1}\right)\right) U_5 \\
=& k_0 f_k\left(m_t^{k,0}, \psi\right) + \frac{1}{2} b_e^2 f_k^2\left(m_t^{k,0}, \psi\right) \\
& + \log c_t(0,0) - \frac{1}{2}(e_t^j)^2
\end{aligned}
$$

Households outside the labour force that choose to participate in the credit market game (since they drew $\psi \leq m_t^{k,0}$) have a very similar ex-ante utility given by:

$$
\begin{aligned}
U_{An} =& q\left(0, e_t^{k,0}\right) U_7 + \left(1 - q\left(1, e_t^{k,1}\right)\right) U_8 \\
=& k_0 f_k\left(m_t^{k,0}, \psi\right) + \frac{1}{2} b_e^2 f_k^2\left(m_t^{k,0}, \psi\right) \\
& + \log c_t(0,0)
\end{aligned}
\tag{2.1}
$$

Households that choose not to participate in the credit market game follow easily from the above.

Note that $k_1$ and $k_0$ are of interest to us later, since part of the motivation of constructing this model is to examine the effects of government pressure on banks to increase credit availability.

### 2.4.2 The job market game

Thus having characterised behaviours in the credit market game, we turn now to the job market game.

#### Households with high credit aversion

Let us assume that $m_t^{k,0} < m_t^{k,1}$, meaning that a greater proportion of employed households than unemployed households will enter the credit market game.[3] Households with high credit aversion, characterised by $\psi > m_t^{k,1}$, will not participate in the credit game irrespective of outcome in the job market game, and thus if they choose to participate in the job market game they seek to maximise:

$$\begin{aligned}E(U) =& p(e_t^j)U_3 + (1-p(e_t^j))U_6\\ =& p(e_t^j)\left(\log\left(c_t(1,0)\right) - F^j - \varsigma_j(1+\sigma_j)\phi^{\sigma_j}\right)\\ &+ \left(1-p(e_t^j)\right)\log\left(c_t(0,0)\right) - \frac{1}{2}(e_t^j)^2\\ =& (\eta + a_e e_t^j)\left(\log\left(\frac{c_t(1,0)}{c_t(0,0)}\right) - F^j - \varsigma_j(1+\sigma_j)\phi^{\sigma_j}\right)\\ &+ \log\left(c_t(0,0)\right) - \frac{1}{2}(e_t^j)^2\end{aligned}$$

This is maximised at (where we define $m_t^{j,h}$ later):

$$\begin{aligned}e_t^{j,h} &= \max\left\{a_e\left(\log\left(\frac{c_t(1,0)}{c_t(0,0)}\right) - F^j - \varsigma_j(1+\sigma_j)\phi^{\sigma_j}\right), 0\right\}\\ &= \max\left\{a_e\varsigma_j(1+\sigma_j)\left((m_t^{j,h})^{\sigma_j} - \phi^{\sigma_j}\right), 0\right\}.\end{aligned}$$

Using the first expression, we get an optimal expected utility of

$$\begin{aligned}E(U) =& \eta\left(\log\left(\frac{c_t(1,0)}{c_t(0,0)}\right) - F^j - \varsigma_j(1+\sigma_j)\phi^{\sigma_j}\right)\\ &+ \frac{1}{2}a_e^2\left(\log\left(\frac{c_t(1,0)}{c_t(0,0)}\right) - F^j - \varsigma_j(1+\sigma_j)\phi^{\sigma_j}\right)^2 + \log\left(c_t(0,0)\right)\end{aligned}$$

Households that do not participate in the job market game simply obtain $U_9 = \log(c_t(0,0))$, and this is thus their certain (expected) utility. Thus, the marginal household with $\phi = m_t^{j,h}$ will be characterised by:

[3] This assumption seems reasonable, since the uncertainty of being unemployed will discourage households from taking on debt.

$$\log\left(\frac{c_t(1,0)}{c_t(0,0)}\right) = F^j + \varsigma_j(1+\sigma_j)\left(m_t^{j,h}\right)^{\sigma_j}.$$

Using this, we can rewrite the optimal expected utility of households with high credit aversion that choose to participate in the job market as:

$$U_H = \eta f_j\left(m_t^{j,h},\phi\right) + \frac{1}{2}a_e^2 f_j^2\left(m_t^{j,h},\phi\right) + \log\left(c_t(0,0)\right) \tag{2.2}$$

with $f_j(m,\phi) = \varsigma_j(1+\sigma_j)\left(m^{\sigma_j} - \phi^{\sigma_j}\right)$.

**Households with medium credit aversion**

Households with $m_t^{k,0} < \psi \leq m_t^{k,1}$ will choose to enter the credit market game if employed, but will stay out if unemployed. Thus, if they choose to enter the job market game, they will maximise:

$$\begin{aligned}
E(U) =& p(e_t^j)\left(U_{A1}\right) + \left(1 - p(e_t^j)\right)U_6 \\
=& p(e_t^j)\left(U_{A1} - U_6\right) + U_6 \\
=& p(e_t^j)\left[k_1 f_k\left(m_t^{k,1},\psi\right) + \frac{1}{2}b_e^2 f_k^2\left(m_t^{k,1},\psi\right)\right] \\
&+ p(e_t^j)\left[\log\left(\frac{c_t(1,0)}{c_t(0,0)}\right) - F^j - \varsigma_j(1+\sigma_j)\phi^{\sigma_j}\right] \\
&+ \log\left(c_t(0,0)\right) - \frac{1}{2}(e_t^j)^2 \\
=& p(e_t^j)\left[k_1 f_k\left(m_t^{k,1},\psi\right) + \frac{1}{2}b_e^2 f_k^2\left(m_t^{k,1},\psi\right) + f_j\left(m_t^{j,h},\phi\right)\right] \\
&+ \log\left(c_t(0,0)\right) - \frac{1}{2}(e_t^j)^2
\end{aligned}$$

Which is maximised at

$$\begin{aligned}
e_t^{j,m} &= \max\left\{a_e\left[k_1 f_k\left(m_t^{k,1},\psi\right) + \frac{1}{2}b_e^2 f_k^2\left(m_t^{k,1},\psi\right) + f_j\left(m_t^{j,h},\phi\right)\right], 0\right\} \\
&= \max\left\{a_e\left[k_1 f_k\left(m_t^{k,1},\psi\right) + \frac{1}{2}b_e^2 f_k^2\left(m_t^{k,1},\psi\right)\right] + e_t^{j,h}, 0\right\}.
\end{aligned}$$

Note the nice relationship between effort levels obtained here: two households with identical aversion to work ($\psi$) will, dependent on the benefit they accrue later from their credit aversion parameters, choose different effort levels. Additionally, we see that as we decrease $\phi$, we maintain piecewise continuity of optimal effort levels, even as we transition from high credit aversion to low.

Households with medium credit aversion that choose to enter the job market game thus obtain an optimal expected utility of the form:

$$\eta\left[k_1 f_k\left(m_t^{k,1},\psi\right)+\frac{1}{2}b_e^2 f_k^2\left(m_t^{k,1},\psi\right)+f_j\left(m_t^{j,h},\phi\right)\right]$$
$$+\frac{1}{2}a_e^2\left[k_1 f_k\left(m_t^{k,1},\psi\right)+\frac{1}{2}b_e^2 f_k^2\left(m_t^{k,1},\psi\right)+f_j\left(m_t^{j,h},\phi\right)\right]^2+\log\left(c_t(0,0)\right)$$

In this case, as above, the marginal household will be characterised by equality of the above utility with the utility from not participating in the job market game, which is simply $U_9 = \log\left(c_t(0,0)\right)$. An equivalent condition for this is thus:

$$k_1 f_k\left(m_t^{k,1},\psi\right)+\frac{1}{2}b_e^2 f_k^2\left(m_t^{k,1},\psi\right)+f_j\left(m_t^{j,h},m_t^{j,m}\right)=0$$

Which leads us to the following relation:

$$\varsigma_j(1+\sigma_j)(m_t^{j,m})^{\sigma_j}=\varsigma_j(1+\sigma_j)(m_t^{j,h})^{\sigma_j}+k_1 f_k\left(m_t^{k,1},\psi\right)+\frac{1}{2}b_e^2 f_k^2\left(m_t^{k,1},\psi\right)$$

Note that this expression for the marginal household in the job market is also piecewise continuous, that is for $\psi = m_t^{k,1}$, then $m_t^{j,m} = m_t^{j,h}$. Note also the inverse relationship between $\psi$ and $m_t^{j,m}$, as hinted at previously regarding the optimal effort levels. This is exactly as we might expect: A bigger upside from participation (the chance to become optimising) will motivate more people to join the job market: households with high $\phi$ that might not otherwise join the job market may do so if they have sufficiently low $\psi$ that the benefits from access to the credit market vastly outweigh the costs.

We use the above definition of $m_t^{j,m}$ (by subtracting $\varsigma_j(1+\sigma_j)\psi^{\sigma_j}$ from both sides) to rewrite the optimal expected utility for households with medium credit aversion participating in the job market game as:

$$U_M = \eta f_j\left(m_t^{j,m},\phi\right)+\frac{1}{2}a_e^2 f_j^2\left(m_t^{j,m},\phi\right)+\log\left(c_t(0,0)\right) \tag{2.3}$$

**Households with low credit aversion**

Finally, households with $\psi \leq m_t^{k,0}$, will join the credit market irrespective of outcome in the job market. Thus, if they participate in the job market game they will maximise their expected utility, given by:

$$
\begin{aligned}
&p(e_t^j)U_{A1} + \left(1 - p(e_t^j)\right) U_{A0} \\
=&p(e_t^j)(U_{A1} - U_{A0}) + U_{A0} \\
=&p(e_t^j) \left[k_1 f_k \left(m_t^{k,1}, \psi\right) - k_0 f_k \left(m_t^{k,0}, \psi\right) + \frac{1}{2} b_e^2 f_k^2 \left(m_t^{k,1}, m_t^{k,0}\right)\right] \\
&+ p(e_t^j) \left[\log\left(\frac{c_t(1,0)}{c_t(0,0)}\right) - F^j - \varsigma_j(1+\sigma_j)\phi^{\sigma_j}\right] \\
&+ k_0 f_k \left(m_t^{k,0}, \psi\right) + \frac{1}{2} b_e^2 f_k^2 \left(m_t^{k,0}, \psi\right) + \log c_t(0,0) - \frac{1}{2}(e_t^j)^2 \\
=&p(e_t^j) \left[k_1 f_k \left(m_t^{k,1}, \psi\right) - k_0 f_k \left(m_t^{k,0}, \psi\right) + \frac{1}{2} b_e^2 f_k^2 \left(m_t^{k,1}, m_t^{k,0}\right) + f_j \left(m_t^{j,h}, \phi\right)\right] \\
&+ k_0 f_k \left(m_t^{k,0}, \psi\right) + \frac{1}{2} b_e^2 f_k^2 \left(m_t^{k,0}, \psi\right) + \log c_t(0,0) - \frac{1}{2}(e_t^j)^2
\end{aligned}
$$

Which is maximised at

$$
\begin{aligned}
e_t^{j,l} = \max\Bigg\{ a_e \Bigg[ & k_1 f_k \left(m_t^{k,1}, \psi\right) - k_0 f_k \left(m_t^{k,0}, \psi\right) \\
& + \frac{1}{2} b_e^2 f_k^2 \left(m_t^{k,1}, m_t^{k,0}\right) + f_j \left(m_t^{j,h}, \phi\right) \Bigg], 0 \Bigg\}
\end{aligned}
$$

This leads to an optimal expected utility of

$$
\begin{aligned}
&\eta \left[k_1 f_k \left(m_t^{k,1}, \psi\right) - k_0 f_k \left(m_t^{k,0}, \psi\right) + \frac{1}{2} b_e^2 f_k^2 \left(m_t^{k,1}, m_t^{k,0}\right) + f_j \left(m_t^{j,h}, \phi\right)\right] \\
&+ \frac{1}{2} a_e^2 \left[k_1 f_k \left(m_t^{k,1}, \psi\right) - k_0 f_k \left(m_t^{k,0}, \psi\right) + \frac{1}{2} b_e^2 f_k^2 \left(m_t^{k,1}, m_t^{k,0}\right) + f_j \left(m_t^{j,h}, \phi\right)\right]^2 \\
&+ k_0 f_k \left(m_t^{k,0}, \psi\right) + \frac{1}{2} b_e^2 f_k^2 \left(m_t^{k,0}, \psi\right) + \log c_t(0,0)
\end{aligned}
$$

If these households choose not to participate in the job market game their utility will be given by $U_{An}$, calculated previously. Note that $U_{An}$ is in fact the last line of the optimal expected utility above, so an equivalent condition for equality must thus be:

$$
k_1 f_k \left(m_t^{k,1}, \psi\right) - k_0 f_k \left(m_t^{k,0}, \psi\right) + \frac{1}{2} b_e^2 f_k^2 \left(m_t^{k,1}, m_t^{k,0}\right) + f_j \left(m_t^{j,h}, m_t^{j,l}\right) = 0
$$

Which leads us to the following relation:

$$
\begin{aligned}
\varsigma_j(1+\sigma_j)(m_t^{j,l})^{\sigma_j} =& k_1 f_k \left(m_t^{k,1}, \psi\right) - k_0 f_k \left(m_t^{k,0}, \psi\right) \\
& + \frac{1}{2} b_e^2 f_k^2 \left(m_t^{k,1}, m_t^{k,0}\right) + \varsigma_j(1+\sigma_j)(m_t^{j,h})^{\sigma_j}
\end{aligned}
$$

Note again that for $\psi = m_t^{k,0}$, then $m_t^{j,m} = m_t^{j,l}$, and if we add and subtract $\frac{1}{2}b_e^2\varsigma_k(1+\sigma_k)\psi$ on the right side, we can rewrite as:

$$\varsigma_j(1+\sigma_j)(m_t^{j,l})^{\sigma_j} = \varsigma_j(1+\sigma_j)(m_t^{j,m})^{\sigma_j} - k_0 f_k\left(m_t^{k,0},\psi\right) - \frac{1}{2}b_e^2 f_k^2\left(m_t^{k,0},\psi\right)$$

The expression above is also quite intuitive: Since staying out of the job market game is more attractive for households with low credit aversion (because they will join the credit market game anyway and possibly become optimising), less households (compared to households with medium credit aversion) will join the job market game.

Using this expression, we can rewrite the optimal expected utility as:

$$\begin{aligned} U_L = &\eta f_j\left(m_t^{j,l},\phi\right) + \frac{1}{2}a_e^2 f_j^2\left(m_t^{j,l},\phi\right) \\ &+ k_0 f_k\left(m_t^{k,0},\psi\right) + \frac{1}{2}b_e^2 f_k^2\left(m_t^{k,0},\psi\right) + \log c_t(0,0) \end{aligned} \tag{2.4}$$

which in fact is also piecewise continuous with $U_M$ in $\psi = m_t^{k,0}$.

## 2.5 Summary

Households populate the unit square, and split into six different types at the outset, which determines their strategy regarding participation in the games. The situation has been represented in Figure 2.1, and is summarised below:

Households with $\psi > m_t^{k,1}$ and $\phi > m_t^{j,h}$ do not participate in either game, and have the certain ex-ante utility of $\log c_t(0,0)$. Households with the same high credit aversion, but $\phi \leq m_t^{j,h}$ participate in the job market game with no intention of participating in the credit market game. They obtain $U_H$ as their ex-ante utility, as defined in (2.2).

Households with medium credit aversion and $\phi > m_t^{j,m}$ also do not participate in either game, and have the certain ex-ante utility of $\log c_t(0,0)$. Medium credit aversion households with $\phi \leq m_t^{j,m}$ participate in the job market, and will then participate in the credit market game if they obtain a job. Their ex-ante utility is $U_M$, as defined in (2.3).

Households with low credit aversion, that is $\psi \leq m_t^{k,0}$ always participate in the credit market game, so of these, households with $\phi > m_t^{j,h}$ choose not to participate in the job market game and obtain ex-ante utility of $U_{An}$, given in (2.1). And households with $\phi \leq m_t^{j,h}$ participate in the job market game and obtain $U_L$ as their ex-ante utility, given in (2.4).

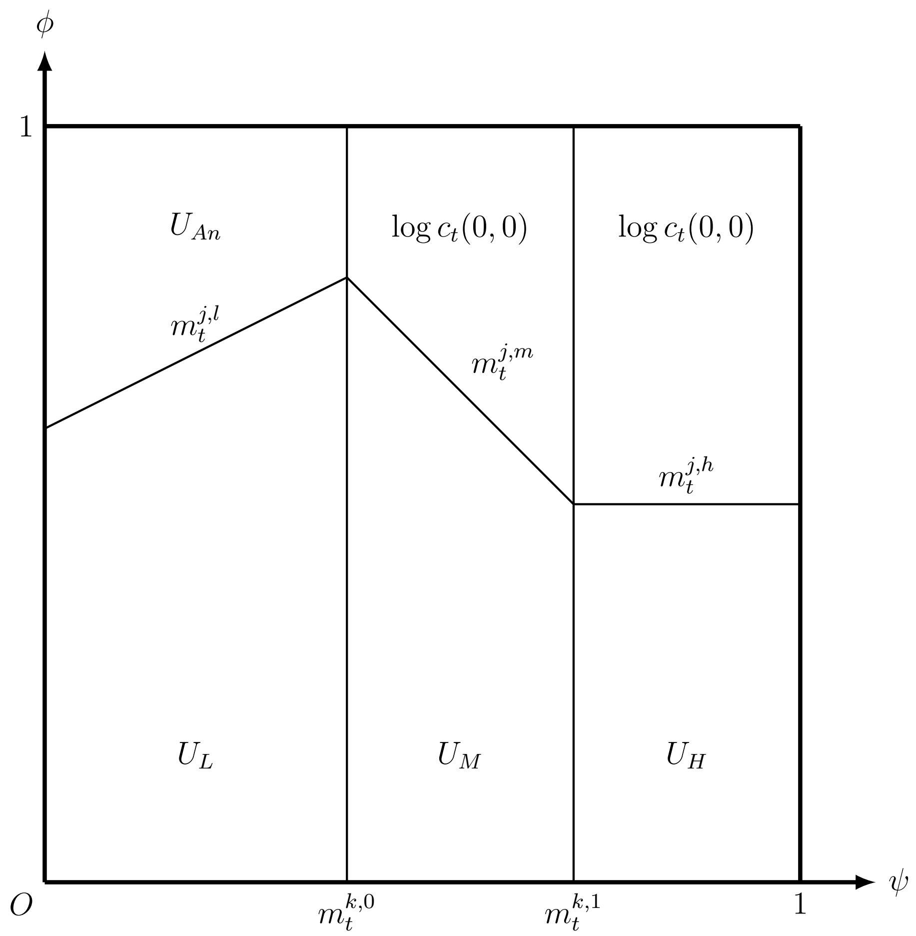


Figure 2.1: A sketch showing the partitioning of the unit square of households according to their strategy and resulting ex-ante utility.

# Chapter 3

# The Family

## 3.1 Introduction

In our model, all households are members of the family, and all interaction with the rest of the economy happens through the family. One of the key issues to address when dealing with a family is to model its behaviour consistently and realistically. Several possible approaches present themselves, ranging from outright coercion of individual households to work (which is incompatible with our previous assumptions regarding households maximising expected utility), across contract theoretical specifications, which we have also briefly considered, where the family proposes a set of contracts to households with the aim of maximising a given utility function subject to incentive compatibility for the households, all the way to the other extreme, where we view the family as having no independent agency at all, and see it as simply an emergent structure of the aggregate behaviour of the households. Our model leans mostly towards the latter, although we will later explore a naive contract theoretical model as well, which produces more tractable behaviour at the macroeconomic level.

For the moment, let us consider the family as an emergent structure. Thus, it seems reasonable to assume the family faces the problem of maximising aggregate intertemporal utility, subject to the standard intertemporal budget constraint for the optimising segment, and the standard "hand to mouth" condition on the rule of thumb segment.

But to arrive at a utility function for the family that fully encapsulates the complex behaviour of the households, we will first have to make a few considerations, starting with the composition of the family. To do this, especially to obtain tractable analytic results, we will need to set $\sigma_j = 1$, and eventually also $\sigma_k = 1$.

## 3.2 The composition of the family

The family is formed after the results of the games have been revealed. But by the law of large numbers, we can calculate the composition of the family by aggregating the behaviour of the households.

The size of the employed households with credit access will be all the households that joined and succeeded in the job market game and joined and succeeded in the credit market game:

$$\begin{aligned}H_t^1 = &\int_0^{m_t^{j,l}} \int_0^{m_t^{k,0}} p\left(e_t^{j,l}\right) q\left(1, e_t^{k,1}\right) d\psi d\phi \\ &+ \int_0^{m_t^{j,m}} \int_{m_t^{k,0}}^{m_t^{k,1}} p\left(e_t^{j,m}\right) q\left(1, e_t^{k,1}\right) d\psi d\phi.\end{aligned}$$

This integral is quite complex, and full details of the methodology for evaluating this integral, along with the others following, is provided in the appendix. Explicit evaluation is only possible with the aforementioned restriction $\sigma_j = 1$.

The unemployed households and households outside the labour force with credit access will be:

$$\begin{aligned}Un_t^1 = &\int_0^{m_t^{j,l}} \int_0^{m_t^{k,0}} \left(1 - p\left(e_t^{j,l}\right)\right) q\left(0, e_t^{k,0}\right) d\psi d\phi \\ &+ \int_{m_t^{j,l}}^1 \int_0^{m_t^{k,0}} q\left(0, e_t^{k,0}\right) d\psi d\phi \\ = &\int_0^1 \int_0^{m_t^{k,0}} q\left(0, e_t^{k,0}\right) d\psi d\phi - \int_0^{m_t^{j,l}} \int_0^{m_t^{k,0}} p\left(e_t^{j,l}\right) q\left(0, e_t^{k,0}\right) d\psi d\phi\end{aligned}$$

As above, a trapezoid approximation is necessary if we do not set $\sigma_j = 1$.

The employed rule-of-thumb households will be the households that joined and succeeded in the job market, and either failed or did not join the credit market game:

$$\begin{aligned}H_t^0 = &\int_0^{m_t^{j,l}} \int_0^{m_t^{k,0}} p\left(e_t^{j,l}\right) \left(1 - q\left(1, e_t^{k,1}\right)\right) d\psi d\phi \\ &+ \int_0^{m_t^{j,m}} \int_{m_t^{k,0}}^{m_t^{k,1}} p\left(e_t^{j,m}\right) \left(1 - q\left(1, e_t^{k,1}\right)\right) d\psi d\phi \\ &+ \int_0^{m_t^{j,h}} \int_{m_t^{k,1}}^1 p\left(e_t^{j,h}\right) d\psi d\phi\end{aligned}$$

Or equivalently:

$$\begin{aligned} H_t^0 = & \int_0^{m_t^{j,l}} \int_0^{m_t^{k,0}} p\left(e_t^{j,l}\right) d\psi d\phi \\ & + \int_0^{m_t^{j,m}} \int_{m_t^{k,0}}^{m_t^{k,1}} p\left(e_t^{j,m}\right) d\psi d\phi \\ & + \int_0^{m_t^{j,h}} \int_{m_t^{k,1}}^{1} p\left(e_t^{j,h}\right) d\psi d\phi \\ & - H_t^1 \end{aligned}$$

While the rule of thumb household who are unemployed or outside the labour force can be determined by the identity $Un_t^0 = 1 - H_t^0 - H_t^1 - Un_t^1$, since we are dealing with a partition of the unit square.

As mentioned earlier, some of these integrals can be evaluated explicitly, while the rest can be explicitly integrated with respect to $\phi$, and will then need numerical integration when integrating with respect to $\psi$. Irrespective of the methodology used, the employment levels turn out to be functions of the marginal levels along with the base utility level, $\log c_t(0,0)$. This will be used later.

## 3.3 The family's utility function

At this point, it is worth explaining why only a single family exists in the model, which then cannot redistribute between rule of thumb and optimising members. It seems reasonable, since we have imposed this partition through the middle of the family, to spilt the family completely. But, if we instead wished to have two families, one optimising and one rule of thumb, problems would appear quite quickly. One might imagine situations where the families would compete against each other to try and "dispose" of the unemployed to the other family. But this behaviour would likely require families to be endowed with a separate decision making process, and would thus bring us into some kind of hybrid contract theory, where two "employers" compete for households, with the complication that the optimising family cannot with certainty allow all applicants entry. Or, alternatively, if we try to construct an emergent behaviour for the two families, we run into issues stemming from households not knowing which family they will end up in at the beginning of the period.

### 3.3.1 The internal constraints upon the family

Thus, to simplify things, only one family exists, and its behaviour is a simple aggregate of household behaviours. Under this specification, we can reinterpret the marginal identities obtained before as incentive compatibility constraints upon the consumption levels. We can safely ignore $m_t^{j,m}$ and $m_t^{j,h}$, since their values follow from the values of the three more fundamental marginal relations, which we repeat below:

$$\log\left(\frac{c_t(1,1)}{c_t(1,0)}\right) = F^k + \varsigma_k(1+\sigma_k)\left(m_t^{k,1}\right)^{\sigma_k} \tag{3.1}$$

$$\log\left(\frac{c_t(0,1)}{c_t(0,0)}\right) = F^k + \varsigma_k(1+\sigma_k)\left(m_t^{k,0}\right)^{\sigma_k} \tag{3.2}$$

$$\log\left(\frac{c_t(1,0)}{c_t(0,0)}\right) = F^j + \varsigma_j(1+\sigma_j)\left(m_t^{j,h}\right)^{\sigma_j} \tag{3.3}$$

Additionally, there are the two internal resource constraints, since the family cannot redistribute between optimising and rule of thumb households:

$$\frac{H_t^1}{H_t^1+Un_t^1}c_t(1,1) + \frac{Un_t^1}{H_t^1+Un_t^1}c_t(0,1) = C_t^1 \tag{3.4}$$

$$\frac{H_t^0}{H_t^0+Un_t^0}c_t(1,0) + \frac{Un_t^0}{H_t^0+Un_t^0}c_t(0,0) = C_t^0 \tag{3.5}$$

These expressions can be combined in various ways, note for example that we can chain together (3.1) and (3.3) to obtain:

$$c_t(1,1) = \exp\Bigg(F^k + \varsigma_k(1+\sigma_k)\left(m_t^{k,1}\right)^{\sigma_k} \\ + F^j + \varsigma_j(1+\sigma_j)\left(m_t^{j,h}\right)^{\sigma_j} + \log(c_t(0,0))\Bigg)$$

This can then be inserted into (3.4) along with (3.2) to express $C_t^1$ as a function of four internal variables, namely $m_t^{k,1}, m_t^{k,0}, m_t^{j,h}$ and $\log(c_t(0,0))$.

Recall that part of the family is assumed to be rule of thumb. This means that this family cannot choose arbitrary combinations of $H_t^0, Un_t^0$ and $C_t^0$. For given $W_t^0$ and $P_t$ the family must choose such that $W_t^0 H_t^0 = C_t^0 P_t$. It proves useful to consider the rule of thumb family as choosing $H_t^0$ only, and viewing $Un_t^0$ and $C_t^0$ being determined from the variables $H_t^1, Un_t^1$ and $C_t^1$.

Thus for a given level of the internal marginals, we can find the corresponding level of the external values, $H_t^1, Un_t^1, H_t^0$ and $C_t^1$. These then in turn determine the remaining $C_t^0$ and $Un_t^0$.

### 3.3.2 The utility function as an aggregate

As mentioned earlier, we do not consider the family to be endowed with any separate motivations or goals. Thus to obtain the utility function for the family, we simply integrate the ex-ante utilities of the households. Recall that there are six different types of households at the outset.

The family's collective utility function becomes:

$$\int_0^{m_t^{j,l}} \int_0^{m_t^{k,0}} U_L d\psi d\phi + \int_{m_t^{j,l}}^1 \int_0^{m_t^{k,0}} U_{An} d\psi d\phi$$
$$+ \int_0^{m_t^{j,m}} \int_{m_t^{k,0}}^{m_t^{k,1}} U_M d\psi d\phi + \int_{m_t^{j,m}}^1 \int_{m_t^{k,0}}^{m_t^{k,1}} \log c_t(0,0) d\psi d\phi$$
$$+ \int_0^{m_t^{j,h}} \int_{m_t^{k,1}}^1 U_H d\psi d\phi + \int_{m_t^{j,h}}^1 \int_{m_t^{k,1}}^1 \log c_t(0,0) d\psi d\phi$$

As before, some of these integrals are easy to evaluate, while others are decidedly non-trivial.

As calculated in the appendix, we arrive at the following approximate expression for the utility function as a function of four internal variables, $m_t^{k,1}$, $m_t^{k,0}$, $m_t^{j,h}$ and $\log(c_t(0,0))$:[1]

$$\begin{aligned}
U \approx & (m_t^{k,0}) \left( \frac{\eta\varsigma_j\sigma_j}{2} \left( (m_t^{j,l})^{1+\sigma_j}_{|0} \right) + \frac{1}{4} a_e^2 \varsigma_j^2 \frac{\sigma_j^2(1+\sigma_j)}{1+2\sigma_j} \left( (m_t^{j,l})^{1+2\sigma_j}_{|0} \right) \right) \\
& - m_t^{k,0} \left( \frac{\eta\varsigma_j\sigma_j}{2} \left( (m_t^{j,h})^{1+\sigma_j} \right) + \frac{1}{4} a_e^2 \varsigma_j^2 \frac{\sigma_j^2(1+\sigma_j)}{1+2\sigma_j} \left( (m_t^{j,h})^{1+2\sigma_j} \right) \right) \\
& + m_t^{k,1} \frac{\eta\varsigma_j\sigma_j}{2} \left( (m_t^{j,h})^{1+\sigma_j} + (m_t^{j,m})^{1+\sigma_j}_{|m_t^{k,0}} \right) \\
& + m_t^{k,1} \frac{1}{4} a_e^2 \varsigma_j^2 \frac{\sigma_j^2(1+\sigma_j)}{1+2\sigma_j} \left( (m_t^{j,h})^{1+2\sigma_j} + (m_t^{j,m})^{1+2\sigma_j}_{|m_t^{k,0}} \right) \\
& + k_0 \varsigma_k \sigma_k (m_t^{k,0})^{1+\sigma_k} + \frac{1}{2} b_e^2 \varsigma_k^2 \left( \frac{\sigma_k^2(1+\sigma_k)(m_t^{k,0})^{1+2\sigma_k}}{1+2\sigma_k} \right) \\
& + (1 - m_t^{k,1}) \left( \eta\varsigma_j\sigma_j (m_t^{j,h})^{1+\sigma_j} + \frac{1}{2} a_e^2 \varsigma_j^2 \left( \frac{\sigma_j^2(1+\sigma_j)(m_t^{j,h})^{1+2\sigma_j}}{1+2\sigma_j} \right) \right) \\
& + \log c_t(0,0)
\end{aligned} \tag{3.6}$$

Thus we have obtained a utility function for the family. But ideally we would like to express this as a function of the variables that form part of

[1] This approximation lets $\sigma_j, \sigma_k$ be general, and uses a trapezoid approximation to the integrals.

the general equilibrium, namely $H_t^1, Un_t^1, H_t^0, C_t^1$. Performing this change of variables is the subject of the next section.

### 3.3.3 Reparametrising the utility function

Recall that we have expressed the employment levels in the model as functions of the three simple marginal levels, along with $\log(c_t(0,0))$. If we use the employment levels to substitute into 3.4 and 3.5, we can treat the four external variables $H_t^1, Un_t^1, H_t^0, C_t^1$ as functions of the internal variables, that is:

$$\begin{pmatrix} H_t^1 \\ Un_t^1 \\ H_t^0 \\ C_t^1 \end{pmatrix} = f \begin{pmatrix} m_t^{k,1} \\ m_t^{k,0} \\ m_t^{j,h} \\ \log(c_t(0,0)) \end{pmatrix} \tag{3.7}$$

If we are able to invert the function $f$ somehow, then we can obtain the utility function as a function of the four external variables. Recall that if the Jacobian matrix of the function $f$, $J_f$, has a non-zero determinant in a particular point, then $f$ is invertible in that point, and $(J_f)^{-1} = J_{(f^{-1})}$.

#### Invertibility of $f$

Using the C++ library GiNaC, we can verify that the determinant is generally non-zero. In certain corner points, the matrix does not exist due to division by zero (see below), but in the other corner points, the determinant is non-zero. In the interior, the matrix is computationally complex and extensive analysis has not been attempted, but all indications are that it is invertible across the entire interior. Thus $f$ is invertible, which we would indeed expect: In the same way that certain marginal levels cause a certain level of employment and consumption to come about, then we should be able to consumption and employment levels that cause certain marginal levels to come about.

The matrix $J$ is large and cumbersome to handle, taking several hours to invert by computer in full generality. Either we must find a simple corner point, where certain effects disappear, or else we insert numerical values for our parameters already at this early stage. The final option that presents itself is to simplify the model. This last option is investigated in the next chapter. For now, let us consider the corner points of the model at hand.

Note firstly that if $m_t^{k,0} = m_t^{k,1} = 0$, then no one will join the credit market game, and $C_t^1$ is indeterminate due to division by zero. Thus this will not do. Although $m_t^{j,h}$ can be zero as long as $m_t^{k,1} > 0$, since even though no households with high credit aversion join the labour market, some households with medium and high credit aversion will (that is $m_t^{j,m}$ and $m_t^{j,h}$ are both positive), so we get $H_t^0 > 0$.

Indeed, choosing for example the point

$$\begin{pmatrix} m_t^{k,1} \\ m_t^{k,0} \\ m_t^{j,h} \\ \log c_t(0,0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and setting $\sigma_k = \sigma_j = 1$ to avoid division by zero when inserting this point into the differentials, the matrix is much more manageable and can be inverted by use of a well-chosen algorithm, which will be explained in the next section.

### Inverting the Jacobian

To invert $J_f$, we use the Cayley-Hamilton formula given by:

$$\begin{aligned}(\det J_f)\, J_f^{-1} = &\left(\frac{\operatorname{Tr}^3 J_f - 3 \operatorname{Tr} J_f \operatorname{Tr} J_f^2 + 2 \operatorname{Tr} J_f^3}{6}\right) I \\ &- \left(\frac{\operatorname{Tr}^2 J_f - \operatorname{Tr} J_f^2}{2}\right) J_f + (\operatorname{Tr} J_f)\, J_f - J_f^3\end{aligned}$$

As mentioned above though, this is only computationally possible in specific points. But once the inverse is obtained, the inverse function theorem can be invoked, and a rough Taylor approximation to the inverse can be constructed. Using $m_t^{k,1}$ as an example, for a point $x = f(m) \in \left(H_t^1, Un_t^1, H_t^0, C_t^1\right)^T$, where $\bar{m}$ is the point in which the matrix is inverted, we get an approximate inverse given by:

$$m_t^{k,1}(x) \equiv m_t^{k,1}(f(\bar{m})) + \begin{pmatrix} \frac{\partial m_t^{k,1}}{\partial H_t^1} \\ \frac{\partial m_t^{k,1}}{\partial Un_t^1} \\ \frac{\partial m_t^{k,1}}{\partial H_t^0} \\ \frac{\partial m_t^{k,1}}{\partial C_t^1} \end{pmatrix}^T (x - f(\bar{m}))$$

Where for notational convenience we omitted the fact that the derivatives are evaluated in the point $f(\bar{m})$. Note that the derivatives are obtained directly from $J_f^{-1}$. Other choices of points for the inversion point of the Jacobian can also be chosen, and rational points in the interior can also be inverted in reasonable time. Each of these will then produce different linear approximations to the marginals.

These expressions can then be substituted into the utility function to re-express it as a function of the external variables. But the utility function will not be separable, since $J_f^{-1}$ generally only has non-zero entries.

For the point considered in the previous section, we can write out these approximate inverse functions, but unfortunately, the linear nature of the approximation means that the inverse functions are very sensitive, easily producing values for the marginals outside the permissible ranges of $[0,1]$. This is due to the fact that three out of four variables on each side are assumed to be within zero to one, while the last one in each case can take on arbitrary values. This in turn means that the first order conditions derived from the utility function will not be valid if they produce invalid internal values. To solve this, the point $\bar{m}$ must be chosen both so that the matrix is tractable, and so that the errors resulting from the first-order Taylor approximation above are kept under control.

## 3.4 The maximisation problem for the family

Assuming the problems above regarding choosing a point $\bar{m}$ could be solved, we can consider the macroeconomic behaviour of the family. Thus we assume the family seeks to maximise expected discounted utility:

$$\begin{aligned}
&\max \left\{ E_t \sum_{t=0}^{\infty} \beta^t U\left(C_t^1, C_t^0, H_t^1, H_t^0\right) \right\} \\
&\text{s.t. } P_t C_t^1 + Q_t B_t \leq B_{t-1} \frac{H_{t-1}^1 + U n_{t-1}^1}{H_t^1 + U n_t^1} + W_t^1 \frac{H_t^1}{H_t^1 + U n_t^1} \\
&\qquad P_t C_t^0 = W_t^0 \frac{H_t^0}{H_t^0 + U n_t^0},
\end{aligned}$$

where $\beta \in ]0,1[$ is the fixed intertemporal discount factor for the family.[2]

To maintain consistency at this point, we interpret $C_t^1$ and $C_t^0$ as per capita consumption. Thus labour income in both cases also has to be per capita, and due to the size of the segments not being one, and possibly

[2]It is interesting to consider how this value should be set, since the family is an aggregate of households, where each household may be optimising one period and rule of thumb the next.

varying over time, we gain the additional complication with the fraction and per capita bond holdings needing to be adjusted from one period to the next.

Several complications present themselves very quickly though. Refer back to (3.6), and note how each place one of the four internal variables is listed, we would need to substitute in a linear combination of the four external variables. This means that the partial derivatives in this general model are too long to be of much use, preventing easy analysis and making closing the DSGE model very complicated. This, along with the reasons outlined above regarding the matrix $J_f$ motivate the next chapter where we develop a reduced model.

# Chapter 4

# A Reduced Model

## 4.1 Introduction

Analysing the model outlined above, with four internal independent variables, proves to be computationally complex. The Jacobian matrix especially is impossible to invert in full generality[1], meaning that obtaining a satisfactory reparametrisation of the utility function is very difficult. Additionally, having each internal variable dependent on each of the four external variables means that the first-order conditions for the family are impractically large. Instead, we here develop a reduced model by removing the possibility of obtaining credit when unemployed.

The reduction in size of the model, and the dimensional reduction of the Jacobian Matrix, means that inversion is possible, and the analysis becomes significantly clearer, but also means that we lose one of the interesting aspects of the model, namely whether the government should pressure banks to favour certain types of households in evaluating loan applications, since now, only employed households may obtain credit.

## 4.2 From six strategies to four

In regards to the formal development of this reduced model, we achieve the effect outlined above by fixing $m_t^{k,0} = 0$. Additionally, we set $\sigma_k = \sigma_j = 1$ for computational and analytical tractability. This further means that $Un_t^0 = 0$, and the two remaining independent employment levels can be calculated exactly.

[1]At first attempt, the program ran for 16 hours on a 3.4 GHz processor, at which point it ran out of RAM. Use of the Cayley-Hamilton formula did reduce this significantly though.

Thus, the amount of employed households with credit access is:

$$
\begin{aligned}
H_t^1 = & k_1 \left(\eta m_t^{j,h} + a_e^2 \varsigma_j (m_t^{j,h})^2\right) m_t^{k,1} \\
& + \frac{\varsigma_k}{2} \left(\frac{\eta k_1^2}{\varsigma_j} + 2a_e^2 k_1^2 (m_t^{j,h}) + 2b_e^2 \left(\eta m_t^{j,h} + a_e^2 \varsigma_j (m_t^{j,h})^2\right)\right) \left(m_t^{k,1}\right)^2 \\
& + \frac{\varsigma_k^2}{3\varsigma_j} \left(3k_1 b_e^2 \left(\eta + 2a_e^2 \varsigma_j m_t^{j,h}\right) + a_e^2 k_1^3\right) \left(m_t^{k,1}\right)^3 \\
& + \frac{\varsigma_k^3}{2\varsigma_j} \left(b_e^4 \left(\eta + 2a_e^2 \varsigma_j m_t^{j,h}\right) + 2a_e^2 k_1^2 b_e^2\right) \left(m_t^{k,1}\right)^4 \\
& + \frac{a_e^2 \varsigma_k^4 k_1 b_e^4}{\varsigma_j} \left(m_t^{k,1}\right)^5 \\
& + \frac{\varsigma_k^5 a_e^2 b_e^6}{3\varsigma_j} \left(m_t^{k,1}\right)^6
\end{aligned}
$$

And the amount of employed households without credit access is:

$$
\begin{aligned}
H_t^0 = & \eta (m_t^{j,h}) + a_e^2 \varsigma_j (m_t^{j,h})^2 \\
& + \frac{1}{2} \left(2a_e^2 \varsigma_k k_1 (m_t^{j,h}) + \frac{\eta \varsigma_k k_1}{\varsigma_j}\right) (m_t^{k,1})^2 \\
& + \frac{1}{3} \left(\frac{\eta \varsigma_k^2 b_e^2}{\varsigma_j} + 2a_e^2 \varsigma_k^2 b_e^2 (m_t^{j,h}) + \frac{a_e^2 \varsigma_k^2 k_1^2}{\varsigma_j}\right) (m_t^{k,1})^3 \\
& + \frac{1}{2} \frac{a_e^2 \varsigma_k^3 k_1 b_e^2}{\varsigma_j} (m_t^{k,1})^4 \\
& + \frac{1}{5} \frac{a_e^2 \varsigma_k^4 b_e^4}{\varsigma_j} (m_t^{k,1})^5 \\
& - H_t^1
\end{aligned}
$$

Note that, pleasingly, when $m_t^{k,1} = 0$, then $H_t^1 = 0$, and the expression for $H_t^0$ collapses to the one found in [Christiano et al., 2010], namely

$$\eta (m_t^{j,h}) + a_e^2 \varsigma_j (m_t^{j,h})^2.$$

Furthermore, when $m_t^{k,1}$ increases, then $H_t^1$ also increases, for positive choices of parameters. This is due to the nature of $m_t^{j,m}$, which is increasing in $m_t^{k,1}$ for positive choices of parameters over the range of $\psi$ in which it applies.

Additionally, the internal resource constraints of the family simplify to:

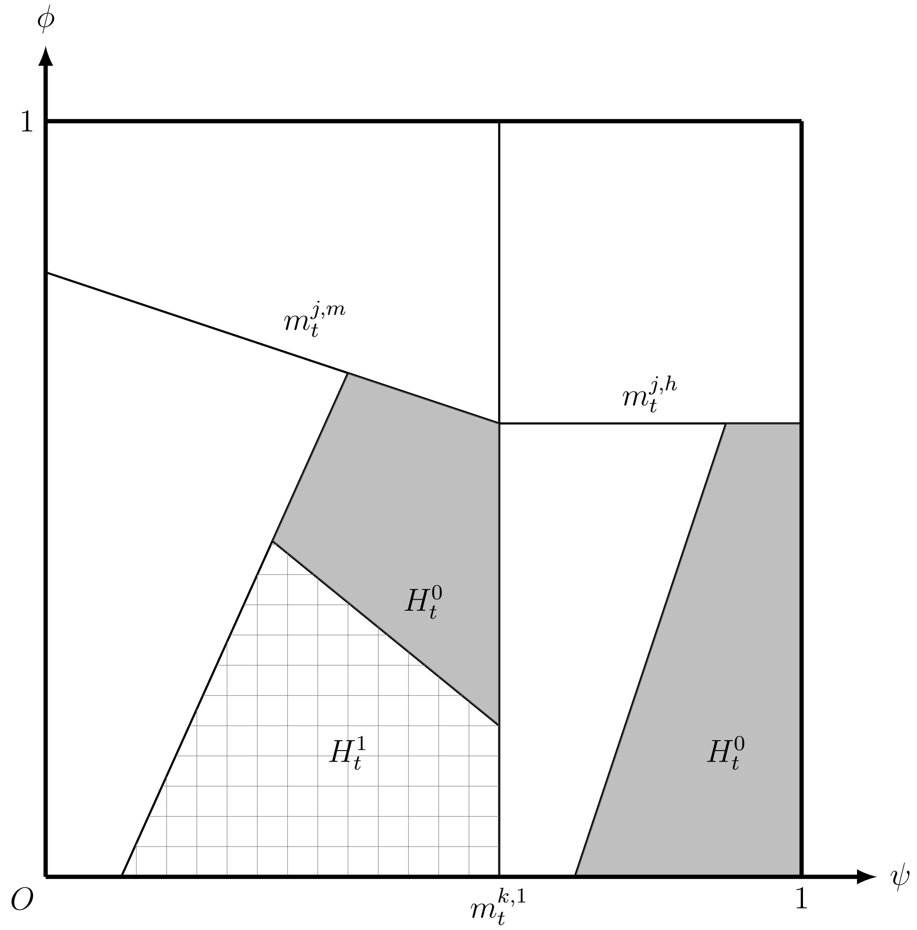


Figure 4.1: A sketch showing the partitioning of the unit square of households according to their outcome.

$$
\begin{aligned}
C_t^1 &= c_t(1,1) \\
C_t^0 &= \frac{H_t^0}{1-H_t^1}c_t(1,0) + \frac{1-H_t^1-H_t^0}{1-H_t^1}c_t(0,0)
\end{aligned}
$$

This partition is simple enough that we can represent it graphically in Figure 4.1, where $H_t^1$ and $H_t^0$ are respectively shaded and hatched. Note that the shape of the regions are not accurate, since we are depicting probabilities across a continuum, but are rather representations of the results arising from households with lower $\phi$ put in a greater effort in the job market. This causes a greater mass of employed households with low aversion to work. Similarly, we get a greater mass of households for lower values of $\psi$ choosing to enter the credit market game.

Note that the figure is drawn such that the probability of obtaining a job even for households with $\phi = 0$ is less than one. It is more difficult though to indicate in this drawing that for example the household characterised by $\phi = \psi = 0$ does indeed join the labour market, likely puts in a large effort, and also puts in a large effort in the job market. This is again due to the fact that we are dealing with continuums and attempting to draw integrals of probabilities.

## 4.3 Obtaining the marginal functions

### 4.3.1 Reducing the size of the problem

In this reduced model, we only have three internal independent variables, namely $m_t^{k,1}, m_t^{j,h}, \log c_t(0,0)$. Additionally, $Un_t^1 = 0$ by definition. So we are left with three external variables of interest: $H_t^1, H_t^0, C_t^1$. This means that we have reduced the size of $J_f$ to a $3 \times 3$ matrix.

Furthermore, note that $H_t^1, H_t^0$ are in fact independent of $\log c_t(0,0)$. This is due to the internal dynamics of the family: households choosing to enter the two games do so in expectation of a reward relative to not participating. Changing the reward to non-participation, $\log c_t(0,0)$, will mean that households will demand a proportionally increased reward to participation in the games, meaning that $\log c_t(1,0)$ and $\log c_t(1,1)$ will have to increase. But this proportional increase is already captured in the definition of $m_t^{j,h}$ and $m_t^{k,1}$ as given in (3.1) and (3.3).

On the other hand, since $C_t^1 = c_t(1,1)$ in this model, and $c_t(1,1)$ is determined by the internal incentive constraints of the family in a simple additive fashion, then changes in $m_t^{k,1}$ and $m_t^{j,h}$ can be dealt with separately from changes in $\log c_t(0,0)$. Thus, we can reduce the problem of reparametrising the dynamics to one of inverting the function

$$f\colon [0,1]^2 \mapsto [0,1]^2$$
$$f\begin{pmatrix} m_t^{k,1} \\ m_t^{j,h} \end{pmatrix} = \begin{pmatrix} H_t^1 \\ H_t^0 \end{pmatrix}$$

and the parametrised function

$$g\colon \mathbb{R} \mapsto \mathbb{R}$$

given by, where lower case letters of the external variables denote log-levels (specifically $c_t^1 = \log C_t^1$):

$$\begin{aligned} c_t^1 =& \log c_t(1,1) \\ =& g\Bigg( \log c_t(0,0) \Bigg) \\ =& F^k + 2\varsigma_k m_t^{k,1} + F_j + 2\varsigma_j m_t^{j,h} - \log c_t(0,0) \end{aligned}$$

Thus we have significantly simplified the model from the previous chapter. Now we need only deal with inverting a $2 \times 2$ matrix.

### 4.3.2 Inverting the Jacobian

So let us begin by examine the function $f$. The Jacobian matrix of this function takes the following form:

$$J_f = \begin{bmatrix} \frac{\partial H_t^1}{\partial m_t^{k,1}} & \frac{\partial H_t^1}{\partial m_t^{j,h}} \\ \frac{\partial H_t^0}{\partial m_t^{k,1}} & \frac{\partial H_t^0}{\partial m_t^{j,h}} \end{bmatrix}$$

The matrix is computationally tractable in full generality, and in general, $\det J \neq 0$. But again, the matrix itself and its determinant are very long (the entries of the matrix are included in the appendix). It is worth noting the value of the determinant along one of the edge cases:

$$\det(J_f)_{|m_t^{k,1}=0} = \frac{3\varsigma_j^2(m_t^{j,h})^2\eta a_e^2 k_1 + 2\varsigma_j^3(m_t^{j,h})^3 a_e^4 k_1 + \varsigma_j(m_t^{j,h})\eta^2 k_1}{\varsigma_j}$$

So the matrix is singular in at least one point: $m_t^{k,1} = m_t^{j,h} = 0$, but in general, the determinant being a large complex polynomial over the domain $[0,1]^2$, the matrix is non-singular.

Thus by the inverse function theorem, we can write:

$$J_f^{-1}(m) = J_{(f^{-1})}\left(f(m)\right),$$

for a point $m \in \begin{pmatrix} m_t^{k,1} \\ m_t^{j,h} \end{pmatrix}$.

This means that we can find the derivatives of the marginals with respect to the employment levels. The problem though is that, as in the previous chapter, we only know the derivatives in the point $f(m)$, and not having any practical way to find $f^{-1}$, we will need to make do with a Taylor expansion, as described next.

### 4.3.3 Constructing the inverse function

To find the marginals as a function of the external variables, we construct a Taylor approximation as in the previous chapter. Denote by $x$ a point $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \begin{pmatrix} H_t^1 \\ H_t^0 \end{pmatrix}$. Then we can construct a Taylor approximation to the coordinates of the inverse function. Using the first coordinate as an example:

$$m_t^{k,1}(x) \approx m_t^{k,1}(\bar{x}) + \frac{\partial m_t^{k,1}}{\partial H_t^1}(\bar{x})\,(x_1 - \bar{x}_1) + \frac{\partial m_t^{k,1}}{\partial H_t^0}(\bar{x})\,(x_2 - \bar{x}_2)$$

One problem with this method immediately presents itself: Recall that the partial derivatives above are obtained evaluated in the point $f(m)$, which means we are not actually choosing a point $\bar{x}$ for our Taylor expansion, but a point $\bar{m}$. Note the subtle problem here: if we want to choose $\bar{x}$ close to the steady state value of employment levels, we need to find the point $\bar{m}$ which has the property $f(\bar{m}) = \bar{x}$. But to do this easily, we need to know the inverse function analytically, which is exactly what we are unable to do.

Overlooking these problems for now, we can collect constants, and get functions of the form:

$$\begin{aligned} m_t^{k,1} &\approx \kappa_k + \kappa_{k1} H_t^1 + \kappa_{k0} H_t^0 \\ m_t^{j,h} &\approx \kappa_j + \kappa_{j1} H_t^1 + \kappa_{j0} H_t^0 \end{aligned}$$

Where the $\kappa$ coefficients are long and complex expressions too large to write out in full, but later in this thesis, tables are presented for the variations in these coefficients as $k_1$ changes.

It will be useful later to also have the marginals expressed as functions of log-employment. We can do this by using the differentials of $h_t^1, h_t^0$ instead in the Jacobian. In this case, we get even more odious coefficients, but can still invert the matrix and write:

$$\begin{aligned} m_t^{k,1} &\approx \mu_k + \mu_{k1} h_t^1 + \mu_{k0} h_t^0 \\ m_t^{j,h} &\approx \mu_j + \mu_{j1} h_t^1 + \mu_{j0} h_t^0 \end{aligned}$$

Turning now our attention to the other function, $g$, we see that to obtain $\log c_t(0,0)$ as a function of the external variables, we can substitute in our linear inverse to the marginals obtained above, to get:

$$
\begin{aligned}
\log c_t(0,0) =& c_t^1 - F^k - F^j - 2\varsigma_k m_t^{k,1} - 2\varsigma_j m_t^{j,h} \\
\approx& c_t^1 - F^k - F^j - 2\left(\varsigma_k \kappa_k + \varsigma_j \kappa_j\right) \\
& - 2\left(\varsigma_k \kappa_{k1} + \varsigma_j \kappa_{j1}\right) H_t^1 - 2\left(\varsigma_k \kappa_{k0} + \varsigma_j \kappa_{j0}\right) H_t^0
\end{aligned}
$$

or alternatively:

$$
\begin{aligned}
\log c_t(0,0) =& c_t^1 - F^k - F^j - 2\varsigma_k m_t^{k,1} - 2\varsigma_j m_t^{j,h} \\
\approx& c_t^1 - F^k - F^j - 2\left(\varsigma_k \mu_k + \varsigma_j \mu_j\right) \\
& - 2\left(\varsigma_k \mu_{k1} + \varsigma_j \mu_{j1}\right) h_t^1 - 2\left(\varsigma_k \mu_{k0} + \varsigma_j \mu_{j0}\right) h_t^0
\end{aligned}
$$

### 4.3.4 Recovering $C_t^0$

When later considering aggregate production at the macroeconomic level, it will be useful to have an expression for $C_t^0$. This can be obtained by taking the resource constraint of the rule of thumb segment, and plugging in the incentive compatibility constraints upon the consumption levels (noting that $C_t^1 = c_t(1,1)$):

$$
\begin{aligned}
C_t^0 =& \frac{H_t^0}{1-H_t^1} c_t(1,0) + \frac{1-H_t^1-H_t^0}{1-H_t^1} c_t(0,0) \\
=& C_t^1 \left( \frac{H_t^0}{1-H_t^1} e^{-F_k - 2\varsigma_k m_t^{k,1}} + \left(1 - \frac{H_t^0}{1-H_t^1}\right) e^{-F_k - F_j - 2\varsigma_k m_t^{k,1} - 2\varsigma_j m_t^{j,h}} \right) \\
=& C_t^1 e^{-F_k - 2\varsigma_k m_t^{k,1}} \left( \frac{H_t^0}{1-H_t^1} + \left(1 - \frac{H_t^0}{1-H_t^1}\right) e^{-F_j - 2\varsigma_j m_t^{j,h}} \right)
\end{aligned}
$$

Note thus that $C_t^0$ increases one-for-one with $C_t^1$. Thus, inserting the approximate inverse functions, we get:

$$
\begin{aligned}
C_t^0 =& C_t^1 e^{-F_k - 2\varsigma_k\left(\mu_k + \mu_{k1} h_t^1 + \mu_{k0} h_t^0\right)} \\
& \times \left[ \frac{H_t^0}{1-H_t^1} + \left(1 - \frac{H_t^0}{1-H_t^1}\right) e^{-F_j - 2\varsigma_j\left(\mu_j + \mu_{j1} h_t^1 + \mu_{j0} h_t^0\right)} \right] \\
=& C_t^1 \left(e^{-F_k - 2\varsigma_k \mu_k}\right) \left(H_t^1\right)^{-2\varsigma_k \mu_{k1}} \left(H_t^0\right)^{-2\varsigma_k \mu_{k0}} \\
& \times \left[ \frac{H_t^0}{1-H_t^1} + \left(1 - \frac{H_t^0}{1-H_t^1}\right) \left(e^{-F_j - 2\varsigma_j \mu_j}\right) \left(H_t^1\right)^{-2\varsigma_j \mu_{j1}} \left(H_t^0\right)^{-2\varsigma_j \mu_{j0}} \right]
\end{aligned}
$$

Taking logs of the above, we see there will be a problem with the last term of the product. The term represents the effects of employed rule of thumb households having to share with non-working households, and as such we know it is is increasing in $H_t^0$, since more employed households in the rule of thumb segment of the family will mean a greater consumption to be shared out. (Recall that a greater $m_t^{j,h}$ is implied by a greater $H_t^0$, so the exponent will grow as well.) Similarly, it is decreasing in $1 - H_t^1$, which is the size of the rule of thumb segment: more households present means less consumption per household. Note also that for all possible values of $H_t^0$, the term will be positive.[2]

It is certainly possible to simplify this expression further to get an approximate, but tractable expression for log-deviations of $c_t^0$ around steady state. Various attempts have indeed been made by the author, but no satisfactory compromise between accuracy and tractability has been found, so let us simply collect it all into a function $F_{crt}\colon [0,1]^2 \mapsto \mathbb{R}$ and write:

$$C_t^0 \equiv C_t^1 F_{crt}(H_t^1, H_t^0) \tag{4.1}$$

## 4.4 The utility function

As discussed in the previous chapter, several choices regarding the utility function of the family might be made. In this section we shall deduce optimality conditions for two alternative methodologies, respectively the "aggregate utility" approach, and a naive contract theoretical approach.

### 4.4.1 The aggregate approach

In this reduced model, due to reducing the number of strategies in existence, we also reduce the length of the expression, and by the choice of $\sigma_k = \sigma_j = 1$ we can write out an exact expression as a function of the three internal variables:

[2]This is in fact a requirement of our model for positive $F^j$: The average consumption in the rule of thumb family must be higher than the consumption level enjoyed by households without a job.

$$
\begin{aligned}
&U\left(m_t^{k,1}, m_t^{j,h}, \log c_t(0,0)\right) \\
&= \int_0^{m_t^{j,m}} \int_0^{m_t^{k,1}} U_M d\psi d\phi + \int_0^{m_t^{j,h}} \int_{m_t^{k,1}}^1 U_H d\psi d\phi \\
&\quad + \int_{m_t^{j,m}}^1 \int_0^{m_t^{k,1}} \log c_t(0,0) d\psi d\phi + \int_{m_t^{j,h}}^1 \int_{m_t^{k,1}}^1 \log c_t(0,0) d\psi d\phi \\
&= \log c_t(0,0) + \eta\varsigma_j (m_t^{j,h})^2 + \frac{1}{3} a_e^2 \varsigma_j^2 (m_t^{j,h})^3 \\
&\quad + k_1 \varsigma_k \left( \eta m_t^{j,h} + \frac{a_e^2 \varsigma_j}{2} (m_t^{j,h})^2 \right) (m_t^{k,1})^2 \\
&\quad + \frac{1}{3} \left( k_1^2 \varsigma_k^2 \left( \frac{\eta}{\varsigma_j} + a_e^2 m_t^{j,h} \right) + b_e^2 \varsigma_k^2 \left( \eta m_t^{j,h} + \frac{a_e^2 \varsigma_j}{2} (m_t^{j,h})^2 \right) \right) (m_t^{k,1})^3 \\
&\quad + \frac{1}{12} \left( 6 k_1 b_e^2 \varsigma_k^3 \left( \frac{\eta}{\varsigma_j} + a_e^2 m_t^{j,h} \right) + \frac{a_e^2 \varsigma_k^3 k_1^3}{\varsigma_j} \right) (m_t^{k,1})^4 \\
&\quad + \frac{1}{5} \left( b_e^4 \varsigma_k^4 \left( \frac{\eta}{\varsigma_j} + a_e^2 m_t^{j,h} \right) + \frac{a_e^2 \varsigma_k^4 k_1^2 b_e^2}{\varsigma_j} \right) (m_t^{k,1})^5 \\
&\quad + \frac{1}{6} \frac{a_e^2 \varsigma_k^5 k_1 b_e^4}{\varsigma_j} (m_t^{k,1})^6 \\
&\quad + \frac{4}{27} \frac{a_e^2 \varsigma_k^8 b_e^6}{\varsigma_j} (m_t^{k,1})^9
\end{aligned}
$$

Note that the utility function is increasing in all of the internal marginals. This makes sense, since individual households will only participate in the job market game if they have a private benefit from doing so, and likewise for the credit market game.

This utility function, containing a lot of higher-order terms, becomes exceedingly cumbersome to handle and deduce first order conditions for, so we somewhat arbitrarily drop all but the first and third lines of the expression, since the first line deals with households with high credit aversion, and the third line is the simplest line to capture both $k_1$ and $b_e$. Thus we are left with:

$$
\begin{aligned}
U \equiv &\log c_t(0,0) + \eta\varsigma_j (m_t^{j,h})^2 + \frac{1}{3} a_e^2 \varsigma_j^2 (m_t^{j,h})^3 \\
&+ \frac{1}{3} \left( k_1^2 \varsigma_k^2 \left( \frac{\eta}{\varsigma_j} + a_e^2 m_t^{j,h} \right) + b_e^2 \varsigma_k^2 \left( \eta m_t^{j,h} + \frac{a_e^2 \varsigma_j}{2} (m_t^{j,h})^2 \right) \right) (m_t^{k,1})^3
\end{aligned}
$$

Into the above simplified utility function we may then substitute the expressions obtained above for the marginals, so that we obtain a utility function expressed as a function of the three external variables:

$$
\begin{aligned}
&U(C_t^1, H_t^1, H_t^0) \\
={}&\log C_t^1 - F_k - F_j - 2\left(\varsigma_k \kappa_k + \varsigma_j \kappa_j\right) \\
&- 2\left(\varsigma_k \kappa_{k1} + \varsigma_j \kappa_{j1}\right) H_t^1 - 2\left(\varsigma_k \kappa_{k0} + \varsigma_j \kappa_{j0}\right) H_t^0 \\
&+ \eta \varsigma_j \left(\kappa_j + \kappa_{j1} H_t^1 + \kappa_{j0} H_t^0\right)^2 + \frac{1}{3} a_e^2 \varsigma_j^2 \left(\kappa_j + \kappa_{j1} H_t^1 + \kappa_{j0} H_t^0\right)^3 \\
&+ \frac{k_1^2 \varsigma_k^2}{3} \left(\frac{\eta}{\varsigma_j} + a_e^2 m_t^{j,h}\right) \left(\kappa_k + \kappa_{k1} H_t^1 + \kappa_{k0} H_t^0\right)^3 \\
&+ \frac{\eta b_e^2 \varsigma_k^2}{3} \left(\kappa_j + \kappa_{j1} H_t^1 + \kappa_{j0} H_t^0\right) \left(\kappa_k + \kappa_{k1} H_t^1 + \kappa_{k0} H_t^0\right)^3 \\
&+ \frac{a_e^2 b_e^2 \varsigma_k^2 \varsigma_j}{6} \left(\kappa_j + \kappa_{j1} H_t^1 + \kappa_{j0} H_t^0\right)^2 \left(\kappa_k + \kappa_{k1} H_t^1 + \kappa_{k0} H_t^0\right)^3 \\
\equiv{}&\log C_t^1 - Z\left(H_t^1, H_t^0\right)
\end{aligned}
$$

### 4.4.2 A simplifying approach

As previously discussed, alternative modelling approaches to the family could certainly be considered. The main reason for this would be to obtain a simpler $Z(H_t^1, H_t^0)$, so that the first order conditions which are to be considered later take on a simpler form.

As a naive suggestion, suppose the utility function for the family is simply given by:

$$
U(C_t^1, H_t^1, H_t^0) = \log C_t^1 - \frac{\left(H_t^1\right)^{1+\varphi}}{1+\varphi}
$$

This is technically possible, since $C_t^0$ and $H_t^0$ will adjust based on the incentive compatibility inside the family, and the budget constraint for the rule of thumb households. Indeed, there may even be multiple possible choices for the family for $H_t^0$, which reflect internal choices of $m_t^{j,h}, m_t^{k,1}$.

One possible interpretation of the above utility function would be that the family does not care explicitly about the welfare of the rule of thumb households. There is of course a secondary effect, in that by incentive compatibility inside the family, $C_t^0$ will increase when $C_t^1$ does. Likewise, if $H_t^1$ decreases, then $H_t^0$ may also decrease, since less households with medium credit aversion will need to join the job market game for example.

It is not immediately clear whether the optimal choices resulting from this utility function in equilibrium would in fact require the family to somehow dictate effort levels of the households, and thus whether we have completely left the realms of both game theory and contract theory. It seems that, by announcing consumption levels $c_t(0,0), c_t(1,0), c_t(1,1)$ and letting the households react, no coercion is needed. But we overlook verifying this for now, recalling that the main point here was to propose a very simple $Z(H_t^1, H_t^0)$ function to allow for alternative simpler optimality conditions.

## 4.5 Optimality conditions of the family

Dependent on our choice of utility function above, we will of course obtain different optimality conditions. In general, the two utility function specifications do have similarities though, and thus we solve the general problem first.

### 4.5.1 The maximisation problem for the family

It is prudent in this model to be careful regarding the optimality conditions. So they shall be derived using the dynamic programming method in [Tønners, 2013]. Write the problem faced by the family as a recursive value function:

$$
\begin{aligned}
\mathbb{V}(B_{t-1}) &= \max_{C_t^1, H_t^1, H_t^0} \left\{ U\left(C_t^1, H_t^1, H_t^0\right) + \beta E_t \mathbb{V}(B_t) \right\} \\
&= \max_{C_t^1, H_t^1, H_t^0} \left\{ \log C_t^1 - Z(H_t^1, H_t^0) + \beta E_t \mathbb{V}(B_t) \right\}
\end{aligned}
$$

And the budget constraint for the optimising segment becomes, after noting that all members of the optimising household are employed, and by our assumptions supply one unit of labour each causing the $\frac{H_t^1}{H_t^1 + Un_t^1}$ term to reduce to one:

$$
B_t = \frac{1}{Q_t} \left( \frac{B_{t-1} H_{t-1}^1}{H_t^1} + W_t^1 - C_t^1 P_t \right)
$$

### 4.5.2 A dimensional reduction

We use the budget constraint for the rule of thumb segment to express $H_t^0$ in optimum as a function of $C_t^1$ and $H_t^1$. This is convenient, since $H_t^0$ has no influence on the state variable, and means that an optimal path for $C_t^1$ and $H_t^1$ will together imply the path for $H_t^0$.

That is, by inserting the incentive-compatibility relation between $C_t^0$ and $C_t^1$ given in (4.1) into the rule of thumb budget constraint:

$$
\begin{aligned}
H_t^0 = & \frac{P_t C_t^0 \left(1 - H_t^1\right)}{W_t^0} = \frac{P_t C_t^1 \left(1 - H_t^1\right)}{W_t^0} F_{crt}\left(H_t^1, H_t^0\right) \\
= & \frac{P_t C_t^1 \left(1 - H_t^1\right)}{W_t^0} \left(e^{-F_k - 2\varsigma_k \mu_k}\right) \left(H_t^1\right)^{-2\varsigma_k \mu_{k1}} \left(H_t^0\right)^{-2\varsigma_k \mu_{k0}} \\
& \times \left[\frac{H_t^0}{1 - H_t^1} + \left(1 - \frac{H_t^0}{1 - H_t^1}\right) \left(e^{-F_j - 2\varsigma_j \mu_j}\right) \left(H_t^1\right)^{-2\varsigma_j \mu_{j1}} \left(H_t^0\right)^{-2\varsigma_j \mu_{j0}}\right] \\
= & \frac{P_t C_t^1}{W_t^0} \left(e^{-F_k - 2\varsigma_k \mu_k}\right) \left(H_t^1\right)^{-2\varsigma_k \mu_{k1}} \left(H_t^0\right)^{-2\varsigma_k \mu_{k0}} \\
& \times \left[H_t^0 + \left(1 - H_t^1 - H_t^0\right) \left(e^{-F_j - 2\varsigma_j \mu_j}\right) \left(H_t^1\right)^{-2\varsigma_j \mu_{j1}} \left(H_t^0\right)^{-2\varsigma_j \mu_{j0}}\right]
\end{aligned}
$$

Note that for fixed numerical values of $\varsigma_k, \varsigma_j$ and the $\mu$ parameters[3] then, given $C_t^1$ and $H_t^1$, this is simply a polynomial in $H_t^0$, the roots of which can then be found numerically, meaning we can express $H_t^0$ as a function of $C_t^1$ and $H_t^1$:

$$H_t^0 \equiv S(C_t^1, H_t^1).$$

Indeed, even for unknown values of the parameters, we can use the implicit function theorem to obtain the derivatives of $S$, namely $S_{H_t^1}$ and $S_{C_t^1}$, which will be needed later. Occasionally it also proves useful to consider $H_t^1$ and $H_t^0$ as the fundamental variables, and solve for $C_t^1$. In this case, we get:

$$C_t^1 = \frac{W_t^0 H_t^0}{P_t \left(1 - H_t^1\right) F_{crt}\left(H_t^1, H_t^0\right)}$$

For now, by substituting in the above for $H_t^0$ as well as the optimising segment's constraint we arrive at the following maximisation problem in two control variables:

$$\max_{C_t^1, H_t^1} \left\{ \log C_t^1 - Z(H_t^1, S(C_t^1, H_t^1)) + \beta E_t \mathbb{V}\left(\frac{\frac{B_{t-1} H_{t-1}^1}{H_t^1} + W_t^1 - C_t^1 P_t}{Q_t}\right) \right\}$$

[3]Recall though that the $\mu$ parameters may contain $k_1$, so these parameters will be subject to shocks.

### 4.5.3 The optimality conditions

The first order condition of this with respect to $C_t^1$ is thus, where we drop the function arguments for notational convenience:

$$\frac{1}{C_t^1} - Z_{H_t^0} S_{C_t^1} = \beta E_t \mathbb{V}'\left(B_t\right) \frac{P_t}{Q_t} \tag{4.2}$$

and similarly with respect to $H_t^1$:

$$Z_{H_t^1} + Z_{H_t^0} S_{H_t^1} = \beta E_t \mathbb{V}'\left(B_t\right) \frac{B_{t-1} H_{t-1}^1}{Q_t \left(H_t^1\right)^2} \tag{4.3}$$

Combining (4.2) and (4.3), we get the following intratemporal optimality condition:

$$\frac{Z_{H_t^1} + Z_{H_t^0} S_{H_t^1}}{\frac{1}{C_t^1} - Z_{H_t^0} S_{C_t^1}} = \frac{B_{t-1} H_{t-1}^1}{P_t \left(H_t^1\right)^2} \tag{4.4}$$

We see that the intratemporal optimality condition for the family is actually state-dependent, being influenced by the amount of income saved from the previous period. The state-dependence comes about due to the fact that the size of the optimising family is not fixed: If $B_{t-1}H_{t-1}^1$ is large, households know that there is a large additional income available for consumption if they put in the effort and join the optimising family. This is a significant departure from the standard model: state-dependence in this case would usually require a capital stock.

It is worth noting that the optimality condition is not necessarily independent of wage levels. Indeed, $W_t^0$ enters through the $Z$ function. But superficially, the optimality condition is indeed independent of $W_t^1$. This is due to the fact that (as will be shown later) $W_t^1$ adjusts to reflect the relative scarcity of $H_t^0$ with respect to $H_t^1$.

For the intertemporal optimality condition, we employ the envelope theorem to differentiate $\mathbb{V}(B_{t-1})$ with respect to $B_{t-1}$ and get:

$$\mathbb{V}'(B_{t-1}) = \frac{\beta}{Q_t} E_t \mathbb{V}'(B_t)$$

Inserting this into (4.2) we get:

$$\frac{1}{C_t^1} - Z_{H_t^0} S_{C_t^1} = P_t \mathbb{V}'(B_{t-1})$$

Which, forwarded one period gives:

$$\frac{1}{C^1_{t+1}} - Z_{H^0_{t+1}} S_{C^1_{t+1}} = P_{t+1}\mathbb{V}'(B_t)$$

and finally implies an Euler equation of the form:

$$\frac{1}{C^1_t} - Z_{H^0_t} S_{C^1_t} = \beta E_t \left[ \frac{\frac{1}{C^1_{t+1}} - Z_{H^0_{t+1}} S_{C^1_{t+1}}}{P_{t+1}} \right] \frac{P_t}{Q_t}$$

So, due to the presence of rule of thumb consumers in the family, and the internal interaction between the rule of thumb and optimising segments, our Euler equation takes on a more complicated shape than usual. Although using (4.4) we can express the optimal $H^1_t$ at least implicitly as a function of $C^1_t$, and then substitute this into the Euler equation to eliminate employment, bearing in mind that the assumptions of our model put an upper limit on $H^1_t$.

### Explicit results

Note that in our alternative utility function specification above, then $Z_{H^0_t} = 0$, and thus the optimality conditions reduce and become:

$$\begin{aligned} C^1_t \left(H^1_t\right)^{2+\varphi} &= \frac{B_{t-1} H^1_{t-1}}{P_t} \\ \frac{1}{C^1_t} &= \beta E_t \left[ \frac{\frac{1}{C^1_{t+1}}}{P_{t+1}} \right] \frac{P_t}{Q_t} \end{aligned}$$

which in log-linear give the following intratemporal optimality condition:

$$b^1_{t-1} + h^1_{t-1} - p_t = c^1_t + (2+\varphi) h^1_t$$

where $b_t = \log B_t$. Additionally, we get the standard Euler equation:

$$\begin{aligned} c^1_t &= -\log\beta + \log Q_t + E_t[\pi_{t+1}] + E_t[c^1_{t+1}] \\ &= E_t[c^1_{t+1}] - (i_t - E_t[\pi_{t+1}] - \rho) \end{aligned}$$

where $i_t = -\log Q_t$, and $\rho = -\log\beta$.

A short comment is probably due here: Even with this extremely simplified and strongly lacking in micro-foundations utility function, we still preserve the effect of the games upon the behaviour of the economy, since the dynamics of the family, with involuntary unemployment and involuntary credit scarcity, will still apply and play in when we come to consider production and equilibrium in more ways than we see here.

# Chapter 5

# Production

## 5.1 Introduction

In this model, being that focus is very much on the household side of things, we seek to let production be as standard as possible. To whit, we postulate intermediate goods producers subject to Calvo price rigidities each having a Cobb-Douglas production function taking as inputs $H_t^1, H_t^0$, and each side of the family buying a consumption index of these intermediate goods.

## 5.2 The consumption index

Implicitly in the previous chapters, $C_t^1$ and $C_t^0$ were consumption indices of the intermediate goods. Borrowing notation from [Galí, 2008], and noting that the logic remains unchanged in our environment, we arrive at the following definitions.

$$
\begin{aligned}
C_t^1 &= \left(\int_0^1 C_t^1(i)^{1-\frac{1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} \\
C_t^0 &= \left(\int_0^1 C_t^0(i)^{1-\frac{1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} \\
P_t &= \left(\int_0^1 P_t(i)^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}
\end{aligned}
$$

From these consumption indices, we get the set of demand equations:

$$C_t^1(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} C_t^1$$
$$C_t^0(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} C_t^0$$

Thus, by calculating the weighted average of these two demand levels, the demand curve facing the producer of the $i$'th intermediate good is then a weighted average of the two demands, denoted here by $C_t(i)$:

$$\begin{aligned} C_t(i) &= H_t^1 C_t^1(i) + (H_t^0 + Un_t^0) C_t^0(i) \\ &= \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} \left(H_t^1 C_t^1 + (1 - H_t^1) C_t^0\right) \\ &\equiv \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} C_t \end{aligned}$$

Where we have introduced $C_t$ as the weighted average of the two aggregate consumption levels.

We are not overly concerned with the fact that employment enters into the demand curve. The actions of the individual monopolist will not influence aggregate employment, ruling out the possibility of a monopolist choosing to increase production to increase employment in an attempt to shift up the demand curve.

Thus, the demand curve is functionally identical to the one in Gali, in that the firm can only affect $P_t(i)$.

## 5.3 Intermediate goods

A continuum of intermediate goods exists, spread along the unit interval. Each intermediate good is produced by a monopolist, using the following production function:

$$Y_{i,t} = A_t \left(H_{i,t}^1\right)^{\alpha_1} \left(H_{i,t}^0\right)^{\alpha_0}.$$

That is, rule of thumb and optimising households supply different types of labour, which enter into a standard Cobb-Douglas production function. Depending on the sum of $\alpha_1 + \alpha_0$, we can achieve various returns to scale. We shall mainly consider decreasing returns to scale, and when analytically necessary, impose constant returns to scale.

Each intermediate goods monopolist is subject to Calvo price frictions, meaning that only a fraction $1 - \theta$ may adjust their price each period.

### 5.3.1 Aggregate price dynamics

As in [Galí, 2008], $\theta$ becomes an index of price stickiness, and the log-linearised aggregate price dynamics are unchanged in this setup, so they are given by:

$$\pi_t = (1-\theta)(p_t^* - p_{t-1}) \tag{5.1}$$

### 5.3.2 Optimal employment ratios

The intermediate goods monopolist seeks to minimise expenditure given a certain level of production. This implies a relationship between $H_{i,t}^1$ and $H_{i,t}^0$. The Lagrangian for this is:

$$W_t^1 H_{i,t}^1 + W_t^0 H_{i,t}^0 - \lambda \left(Y_{i,t} - A_t \left(H_{i,t}^1\right)^{\alpha_1} \left(H_{i,t}^0\right)^{\alpha_0}\right)$$

Which has first order conditions, after substituting in the constraint:

$$W_t^1 + \lambda\alpha_1 \left(\frac{Y_{i,t}}{H_{i,t}^1}\right) = 0$$
$$W_t^0 + \lambda\alpha_0 \left(\frac{Y_{i,t}}{H_{i,t}^1}\right) = 0$$

Implying (by solving for $\lambda$, equating and rearranging to insert into the constraint) optimal employment levels of:

$$H_{i,t}^1 = \left(\frac{Y_{i,t}}{A_t}\left(\frac{W_t^0\alpha_1}{W_t^1\alpha_0}\right)^{\alpha_0}\right)^{\frac{1}{\alpha_1+\alpha_0}}$$
$$H_{i,t}^0 = \left(\frac{Y_{i,t}}{A_t}\left(\frac{W_t^1\alpha_0}{W_t^0\alpha_1}\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1+\alpha_0}}$$

And a cost function of the form:

$$\begin{aligned}
&\Psi_t\left(W_t^1, W_t^0, Y_{i,t}\right) \\
=&W_t^1\left(\frac{Y_{i,t}}{A_t}\left(\frac{W_t^0\alpha_1}{W_t^1\alpha_0}\right)^{\alpha_0}\right)^{\frac{1}{\alpha_1+\alpha_0}} \\
&+ W_t^0\left(\frac{Y_{i,t}}{A_t}\left(\frac{W_t^1\alpha_0}{W_t^0\alpha_1}\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1+\alpha_0}} \\
=&\left(\frac{Y_{i,t}}{A_t}\right)^{\frac{1}{\alpha_1+\alpha_0}}\left(W_t^1\left(\frac{W_t^0\alpha_1}{W_t^1\alpha_0}\right)^{\frac{\alpha_0}{\alpha_1+\alpha_0}} + W_t^0\left(\frac{W_t^1\alpha_0}{W_t^0\alpha_1}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_0}}\right)
\end{aligned}$$

### 5.3.3 Optimal price setting

Optimal price setting in the above environment means that a firm re-optimising at time $t$ will choose the optimal price $P_t$ which maximises expected discounted profits. That is the firm solves

$$\max_{P_t} \sum_{k=0}^{\infty} E_t\{Q_{t,t+k}(P_t^* Y_{t+k|t} - \Psi_{t+k}(Y_{t+k,t})\}$$

subject to the sequence of demand constraints

$$Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k}.$$

We do need to be careful at this stage regarding the definition of the discount factor $Q_{t,t+k}$ that firms should use. In the standard model, we assume households own the firms, and thus their discount factor applies. In this model, it seems consistent similarly to assume the family owns the firms, and thus it should be the discount factor for the family that applies. Dependent on our assumption about the utility function this is a quite complex object. This further motivates using the simplified utility function. Regardless, we define $Q_{t,t+k}$ from the Euler equation to be:

$$Q_{t,t+k} = \beta^k E_t \left[\frac{\frac{1}{C^1_{t+k}} - Z_{H^0_{t+k}} S_{C^1_{t+k}}}{\frac{1}{C^1_t} - Z_{H^0_t} S_{C^1_t}}\right] \frac{P_t}{P_{t+k}},$$

which significantly simplifies with the simplified utility function.

Being that we have modelled our producers closely on [Galí, 2008], the conclusions regarding optimal price setting remain unchanged. To whit, steady state marginal costs are constant and equal to frictionless marginal costs, and $Q_{t,t+k} = \beta^k$ in steady state. All of this implies that we can obtain the following log-linearised optimal price setting relationship:

$$p_t^* - p_{t-1} = (1 - \beta\theta) \sum_{k=0}^{\infty} (\beta\theta)^k E_t\{mc_{t+k|t} - mc + p_{t+k} - p_{t-1}\} \tag{5.2}$$

### 5.3.4 Marginal cost

Marginal costs come from differentiating the total costs with respect to $Y_{t+k|t}$. We have already developed an expression for nominal $TC$, namely $\Psi$, so let us differentiate this to obtain nominal marginal costs:

$$\frac{\partial\Psi_{t+k|t}}{\partial Y_{t+k|t}} = \frac{Y_{t+k|t}^{\frac{1-\alpha_1-\alpha_0}{\alpha_1+\alpha_0}} A_{t+k}^{\frac{-1}{\alpha_1+\alpha_0}}}{\alpha_1+\alpha_0}\left(W^1_{t+k}\left(\frac{W^0_{t+k}\alpha_1}{W^1_{t+k}\alpha_0}\right)^{\frac{\alpha_0}{\alpha_1+\alpha_0}} + W^0_{t+k}\left(\frac{W^1_{t+k}\alpha_0}{W^0_{t+k}\alpha_1}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_0}}\right)$$

Without further assumptions, this expression is quite cumbersome, thus let us investigate a few simplifying assumptions we might make:

### Common marginal productivities

If we assume $\alpha_1 = \alpha_0 = \alpha$ and $\alpha < \frac{1}{2}$, then nominal marginal costs become:

$$\begin{aligned}\frac{\partial\Psi_{t+k|t}}{\partial Y_{t+k|t}} &= \frac{Y_{t+k|t}^{\frac{1-2\alpha}{2\alpha}} A_{t+k}^{\frac{-1}{2\alpha}}}{2\alpha}\left(W^1_{t+k}\left(\frac{W^0_{t+k}}{W^1_{t+k}}\right)^{\frac{1}{2}} + W^0_{t+k}\left(\frac{W^1_{t+k}}{W^0_{t+k}}\right)^{\frac{1}{2}}\right)\\ &= \frac{Y_{t+k|t}^{\frac{1-2\alpha}{2\alpha}} A_{t+k}^{\frac{-1}{2\alpha}}}{\alpha}\left(\sqrt{W^1_t W^0_t}\right)\end{aligned}$$

And taking logs and dividing by the price level, we get:

$$mc_{t+k|t} = \frac{y_{t+k|t} - a_{t+k}}{2\alpha} - y_{t+k|t} + \frac{w^1_{t+k} + w^0_{t+k}}{2} - p_{t+k} - \log\alpha.$$

Initially, this seems like a very attractive assumption, but it turns out that this assumption is problematic with respect to the internal dynamics of the family. The reason for this is to be found in the optimality condition for the producers in this case implying $W^1_t H^1_t = W^0_t H^0_t$, meaning that the total labour income for the rule of thumb and optimising segments are the same. To square this with the internal dynamics of the family, which require considerably higher consumption levels for the optimising segment due to incentive compatibility, will mean that $H^1_t$ will need to be significantly smaller than $H^0_t$, and $W^1_t$ much higher than $W^0_t$ for optimising households to maintain their higher per capita consumption. A way out would of course be for the optimising family to take out loans to finance the higher consumption level, so we will occasionally note interesting behaviours under common marginal productivities to see how they play out.

### Differing marginal productivities

The alternative to common marginal productivities is of course differing marginal productivities. In addition, we will for parts of the thesis assume constant returns to scale, which will be commented upon later.

For now, simply assume $\alpha_1+\alpha_0 \leq 1$, thus nominal marginal costs become:

$$
\begin{aligned}
&\frac{\partial \Psi_{t+k|t}}{\partial Y_{t+k|t}} \\
=&\frac{Y_{t+k|t}^{\frac{1-\alpha_1-\alpha_0}{\alpha_1+\alpha_0}} A_{t+k}^{\frac{-1}{\alpha_1+\alpha_0}}}{\alpha_1+\alpha_0} \left( W_{t+k}^1 \left( \frac{W_{t+k}^0 (\alpha_1)}{W_{t+k}^1 \alpha_0} \right)^{\frac{\alpha_0}{\alpha_1+\alpha_0}} + W_{t+k}^0 \left( \frac{W_{t+k}^1 \alpha_0}{W_{t+k}^0 (\alpha_1)} \right)^{\frac{\alpha_1}{\alpha_1+\alpha_0}} \right) \\
=&\frac{Y_{t+k|t}^{\frac{1-\alpha_1-\alpha_0}{\alpha_1+\alpha_0}} A_{t+k}^{\frac{-1}{\alpha_1+\alpha_0}}}{\alpha_1+\alpha_0} \left(W_{t+k}^1\right)^{\frac{\alpha_1}{\alpha_1+\alpha_0}} \left(W_{t+k}^0\right)^{\frac{\alpha_0}{\alpha_1+\alpha_0}} \left( \left( \frac{(\alpha_1)}{\alpha_0} \right)^{\frac{\alpha_0}{\alpha_1+\alpha_0}} + \left( \frac{\alpha_0}{(\alpha_1)} \right)^{\frac{\alpha_1}{\alpha_1+\alpha_0}} \right) \\
=&k_{mc} \left( \frac{Y_{t+k|t}^{1-\alpha_1+\alpha_0}}{A_{t+k}} \left(W_{t+k}^1\right)^{\alpha_1} \left(W_{t+k}^0\right)^{\alpha_0} \right)^{\frac{1}{\alpha_1+\alpha_0}}
\end{aligned}
$$

where $k_{mc} = \frac{\left(\frac{\alpha_1}{\alpha_0}\right)^{\frac{\alpha_0}{\alpha_1+\alpha_0}} + \left(\frac{\alpha_0}{\alpha_1}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_0}}}{\alpha_1+\alpha_0}$.

If we divide by the price level and take logs, we get:

$$
mc_{t+k|t} = \log k_{mc} + \frac{y_{t+k|t} - a_{t+k} + \alpha_1 w_{t+k}^1 + \alpha_0 w_{t+k}^0}{\alpha_1+\alpha_0} - y_{t+k|t} - p_{t+k}
$$

### Constant returns to scale

As mentioned above, in parts of the thesis we will further assume constant returns to scale. Thus, we get the simple expression:

$$
mc_{t+k|t} = \alpha_1 w_{t+k}^1 + \alpha_0 w_{t+k}^0 - p_{t+k} - a_{t+k} + \log k_1
$$

where the constant reduces to: $k_{mc} = \left(\frac{\alpha_1}{\alpha_0}\right)^{\alpha_0} + \left(\frac{\alpha_0}{\alpha_1}\right)^{\alpha_1}$.

We note that in the CRS case, marginal costs for the intermediate firm are independent of when the price was last adjusted, thus all firms share the same marginal costs. This means a lot of the dynamics of the standard DSGE model disappear, but due to the complexity of the interaction between optimising households, rule of thumb households, and their respective employment levels, we will need this assumption to keep the scope of the model reasonable.

# Chapter 6

# Equilibrium

## 6.1 Introduction

As discussed in the chapter on production, we shall assume that households with credit access are more productive. Furthermore, we shall employ the reduced model developed in chapter 3. Under these assumptions, we can use the equilibrium and optimality conditions to close our DSGE model.

## 6.2 Clearing in the goods market

Production equals consumption for every intermediate good:

$$Y_{i,t} = H_t^1 C_t^1(i) + (1 - H_t^1) C_t^0(i)$$

Thus aggregate production must equal aggregate consumption, where we silently define $Y_t$ as the weighted index over $Y_{i,t}$. Note first that aggregate output is closely related to $C_t^1$ due to the internal dynamics of the family, where we use the expression for $C_t^0$ from (4.1):

$$\begin{aligned} Y_t =& H_t^1 C_t^1 + (1 - H_t^1) C_t^0 \\ =& H_t^1 C_t^1 + (1 - H_t^1) C_t^1 F_{crt}(H_t^1, H_t^0) \\ \equiv& C_t^1 F_{agr}(H_t^1, H_t^0) \end{aligned}$$

where we defined $F_{agr} \colon \mathbb{R}^2 \mapsto \mathbb{R}$ as a notational convenience that shall be used imminently. We note that

$$F_{agr}(H_t^1, H_t^0) = H_t^1 + (1 - H_t^1) F_{crt}(H_t^1, H_t^0)$$

Note the structure of this, in that each segment of the family has a different weight in determining aggregate production, dependent on how far their consumption is away from that of the optimising employed households.[1] Note also that we have expressed aggregate production as a product of $C_t^1$ multiplied by a function of the employment levels only.

The above condition is often used to re-express the Euler equation in terms of the output gap. In our case, it is technically possible, but we see that we will not be able to escape the influence of the employment levels. An alternative option presents itself, which is to express the Euler equation as a function of employment and technology levels only.

To do this, use the aggregate production function obtained below to write:

$$C_t^1 = \frac{A_t\,(H_t^1)^{\alpha_1}\,(H_t^0)^{\alpha_0}}{F_{agr}(H_t^1, H_t^0)}$$

And, assuming the simplified utility function from previously, we can obtain an Euler equation relating current employment levels with one-period ahead expectations and inflation:

$$\frac{F_{agr}(H_t^1, H_t^0)}{A_t\,(H_t^1)^{\alpha_1}\,(H_t^0)^{\alpha_0}} = \beta E_t\left[\frac{1}{P_{t+1}}\frac{F_{agr}(H_{t+1}^1, H_{t+1}^0)}{A_{t+1}\left(H_{t+1}^1\right)^{\alpha_1}\left(H_{t+1}^0\right)^{\alpha_0}}\right]\frac{P_t}{Q_t}$$

Which, for a suitable choice of log-linearisation around the natural levels of employment[2] gives:

$$\zeta_{h1}\tilde{h}_t^1 + \zeta_{h0}\tilde{h}_t^0 = E_t\left[\zeta_{h1}\tilde{h}_{t+1}^1 + \zeta_{h0}\tilde{h}_{t+1}^0\right] - (i_t - E_t[\pi_{t+1}] - r_t^n) \tag{6.1}$$

## 6.3 Clearing in the labour market

We define total employment for the rule of thumb and optimising segments as:

$$H_t^1 = \int_0^1 H_{i,t}^1 di$$
$$H_t^0 = \int_0^1 H_{i,t}^0 di$$

[1] This is unsurprising, considering the internal behaviour of the family.

[2] See the next chapter for a discussion on their existence.

Take $H_t^1$ as an example. Insert from the optimal employment levels of the intermediate goods producers to get:

$$
\begin{aligned}
H_t^1 &= \int_0^1 \left( \frac{Y_{i,t}}{A_t} \left( \frac{W_t^0(\alpha_1)}{W_t^1 \alpha_0} \right)^{\alpha_0} \right)^{\frac{1}{\alpha_1+\alpha_0}} di \\
&= \left( \frac{W_t^0 \alpha_1}{W_t^1 \alpha_0} \right)^{\frac{\alpha_0}{\alpha_1+\alpha_0}} \int_0^1 \left( \frac{Y_{i,t}}{A_t} \right)^{\frac{1}{\alpha_1+\alpha_0}} di \\
&= \left( \frac{Y_t}{A_t} \right)^{\frac{1}{\alpha_1+\alpha_0}} \left( \frac{W_t^0 \alpha_1}{W_t^1 \alpha_0} \right)^{\frac{\alpha_0}{\alpha_1+\alpha_0}} \int_0^1 \left( \frac{P_t(i)}{P_t} \right)^{\frac{-\varepsilon}{\alpha_1+\alpha_0}} di
\end{aligned}
$$

And, taking logs, we get:

$$
\begin{aligned}
h_t^1 &= \frac{\alpha_0}{\alpha_1+\alpha_0}(w_t^0 - w_t^1) + \frac{y_t - a_t}{\alpha_1+\alpha_0} + \frac{\alpha_0}{\alpha_1+\alpha_0} \ln\left(\frac{\alpha_1}{\alpha_0}\right) - d_t \\
h_t^0 &= \frac{\alpha_1}{\alpha_1+\alpha_0}(w_t^1 - w_t^0) + \frac{y_t - a_t}{\alpha_1+\alpha_0} + \frac{\alpha_1}{\alpha_1+\alpha_0} \ln\left(\frac{\alpha_0}{\alpha_1}\right) - d_t
\end{aligned}
$$

Dropping the $d_t$ term (following Gali, who argues it is small in a region around steady state), and rearranging the second to insert into the first, we get:

$$
\begin{aligned}
h_t^1 =& \frac{\alpha_0}{\alpha_1+\alpha_0} \left( -\frac{\alpha_1+\alpha_0}{\alpha_1} h_t^0 + \frac{y_t - a_t}{\alpha_1} + \ln\left(\frac{\alpha_0}{\alpha_1}\right) \right) \\
&+ \frac{y_t - a_t}{\alpha_1+\alpha_0} + \frac{\alpha_0}{\alpha_1+\alpha_0} \ln\left(\frac{\alpha_1}{\alpha_0}\right) \\
=& \frac{y_t - a_t}{\alpha_1} - \frac{\alpha_0}{\alpha_1} h_t^0
\end{aligned}
$$

If we rearrange this, we get:

$$
y_t = a_t + \alpha_1 h_t^1 + \alpha_0 h_t^0
$$

Note that this implies an approximate aggregate production function of $Y_t = A_t \left(H_t^1\right)^{\alpha_1} \left(H_t^0\right)^{\alpha_0}$, which strongly suggests that aggregate marginal costs will look similar to firm level marginal costs. But a formal derivation of this fact is done below:

## 6.4 Aggregate marginal costs

To obtain aggregate marginal costs, we may integrate firm level marginal costs across $i \in [0,1]$. Thus:

$$\begin{aligned}
&MC_{t+k}\\
&=\int_0^1 \frac{k_{mc}}{P_{t+k}}\left(\frac{Y_{t+k|t}^{1-\alpha_1-\alpha_0}}{A_{t+k}}\left(W_{t+k}^1\right)^{\alpha_1}\left(W_{t+k}^0\right)^{\alpha_0}\right)^{\frac{1}{\alpha_1+\alpha_0}} di\\
&=\frac{k_{mc}}{P_{t+k}Y_{t+k}}\left(\frac{Y_{t+k}}{A_{t+k}}\right)^{\frac{1}{\alpha_1+\alpha_0}}\left(W_{t+k}^1\right)^{\frac{\alpha_1}{\alpha_1+\alpha_0}}\left(W_{t+k}^0\right)^{\frac{\alpha_0}{\alpha_1+\alpha_0}}\int_0^1\left(\frac{P_{t+k|t}}{P_{t+k}}\right)^{\frac{-\varepsilon(1-\alpha_1-\alpha_0)}{\alpha_1+\alpha_0}} di
\end{aligned}$$

Where the second equality follows from the demand schedule under goods market clearing. Thus, taking logs, we get:

$$mc_{t+k} = \ln k_{mc} - y_{t+k} - p_{t+k} + \frac{y_{t+k} - a_{t+k} + (\alpha_1)w_{t+k}^1 + \alpha_0 w_{t+k}^0}{\alpha_1+\alpha_0} + d_{t+k}$$

where $d_{t+k}$, although different from above, can still be assumed to be zero close to steady state.

Thus, by calculating the difference between firm level marginal costs and aggregate marginal costs, we obtain:

$$\begin{aligned}
mc_{t+k|t} =& mc_{t+k} + \frac{1-\alpha_1-\alpha_0}{\alpha_1+\alpha_0}\left(y_{t+k|t} - y_{t+k}\right)\\
=& mc_{t+k} - \frac{\varepsilon(1-\alpha_1-\alpha_0)}{\alpha_1+\alpha_0}\left(p_t^* - p_{t+k}\right)
\end{aligned}$$

where the second equation comes from the demand schedule, since a firm that last adjusted its price at time $t$ chose the price $p_t^*$. This result is the same as the one found in Gali, when we account for the differences in the production function. Note that when we have constant returns to scale, $mc_{t+k|t} = mc_{t+k}$, as expected.

Thus, as in Gali, we can substitute this into the optimal price setting equation, (5.2), and obtain

$$p_t^* - p_{t-1} = (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k E_t\{\Theta \hat{mc}_{t+k} + (p_{t+k} - p_{t-1})\}$$

Where our $\Theta = \frac{\alpha_1+\alpha_0}{\alpha_1+\alpha_0+\varepsilon(1-\alpha_1-\alpha_0)}$ is different from Gali due to the different production function. Note that $\Theta < 1$ as long as $\alpha_1 + \alpha_0 < 1$, that is the total returns to scale are decreasing.

When we have CRS meanwhile, the marginal costs drop out completely, since $mc_{t+k} = mc$, the frictionless marginal cost.

The difference equation $p_t^* - p_{t-1} = \beta\theta E_t(p_{t+1}^* - p_t) + (1-\beta\theta)\Theta \hat{mc}_t + \pi_t$ gives an equivalent representation, which can be inserted into the equation for the development of the price level, (5.1), to get:

$$\pi_t = \beta E_t\{\pi_{t+1}\} + \lambda \hat{mc}_t \tag{6.2}$$

where $\lambda = \frac{(1-\theta)(1-\beta\theta)}{\theta}\Theta$. Meanwhile, under CRS, the equation becomes almost trivial:

$$\pi_t = \beta E_t\{\pi_{t+1}\}$$

## 6.5 Obtaining the NKPC

Rewrite the expression for aggregate marginal costs:

$$mc_t = \log k_{mc} - y_t - p_t + \frac{y_t - a_t + \alpha_1 w_t^1 + \alpha_0 w_t^0}{\alpha_1 + \alpha_0}$$

To obtain a NKPC for our model, we will need to be very careful, considering the state-dependency of our intratemporal optimality conditions. Thus we start by inserting $w_t^1 = w_t^0 + h_t^0 - h_t^1 + \log\left(\frac{\alpha_1}{\alpha_0}\right)$, which is implied by the first order conditions of the aggregate production function:

$$mc_t = \log k_{mc} - y_t + w_t^0 - p_t + \frac{y_t - a_t + \alpha_1\left(h_t^0 - h_t^1 + \log\left(\frac{\alpha_1}{\alpha_0}\right)\right)}{\alpha_1 + \alpha_0}$$

Substitute out the $y_t$ in the fraction using the production function:

$$\begin{aligned} mc_t =& \log k_{mc} - y_t + w_t^0 - p_t + \frac{\alpha_0 h_t^0 + \alpha_1 \left( h_t^0 + \log\left(\frac{\alpha_1}{\alpha_0}\right)\right)}{\alpha_1 + \alpha_0} \\ =& \log k_{mc} - y_t + w_t^0 - p_t + h_t^0 + \frac{\alpha_1 \log\left(\frac{\alpha_1}{\alpha_0}\right)}{\alpha_1 + \alpha_0} \\ =& \log k_{mc} - y_t + c_t^0 + \log(1 - H_t^1) + \frac{\alpha_1 \log\left(\frac{\alpha_1}{\alpha_0}\right)}{\alpha_1 + \alpha_0} \\ =& -\log \alpha_0 + \log\left(\frac{(1 - H_t^1) F_{crt}(H_t^1, H_t^0)}{F_{agr}(H_t^1, H_t^0)}\right) \end{aligned}$$

Where the penultimate equality follows from the behaviour of the rule of thumb households, and the final equation uses the previously established relationships between respectively $Y_t$ and $C_t^0$ with $C_t^1$, and collects constants.

As in Gali, marginal costs will be constant under flexible prices. But to obtain the same log-deviation of marginal costs away from steady state, we need to be sure that $H_t^1$ and $H_t^0$ have well-defined equilibrium values under flexible prices. This will be investigated in the next chapter. If they do, we can express deviations in marginal cost as a function of deviations from "natural" levels of the employment levels. This means we can write, for a suitable choice of log-linearisation:

$$\hat{mc}_t = \eta_{h1} \tilde{h}_t^1 + \eta_{h0} \tilde{h}_t^0$$

At this stage, we can insert into (6.2) and get:

$$\pi_t = \beta E_t[\pi_{t+1}] - \lambda \left( \eta_{h1} \tilde{h}_t^1 + \eta_{h0} \tilde{h}_t^0 \right) \tag{6.3}$$

Thus, we have managed to express both the NKPC (6.3) and the Euler equation (6.1) in terms of the same variables, namely current and future employment levels, but it still remains to conclude whether our dynamic system has two free variables, inflation and $\tilde{h}_t^1$, or whether, due to the state dependency of the optimality conditions, we will also need to include $\tilde{h}_t^0$. Being that we have not yet used the intratemporal optimality condition it seems likely that once we invoke this, $\tilde{h}_t^1$ will be sufficient, and the rest will follow from this.

Unfortunately, due to the late stage at which a major mistake in the budget constraints was discovered, no attempt has been made to pursue this further. We will briefly comment on the relation monetary policy has on the model though:

## 6.6 The interest rates

For the sake of convenience, the Euler equation is repeated below. Note how the nominal interest rate and the natural interest rate feature:

$$\zeta_{h1}\tilde{h}_t^1 + \zeta_{h0}\tilde{h}_t^0 = E_t\left[\zeta_{h1}\tilde{h}_{t+1}^1 + \zeta_{h0}\tilde{h}_{t+1}^0\right] - (i_t - E_t[\pi_{t+1}] - r_t^n)$$

In our model, we can thus define the natural rate of interest as:

$$r_t^n = \rho + E_t[\triangle c_{t+1}^{1,n}]$$

in this case, we as previously need to check whether $c_t^{1,n}$ has a well-defined relationship with $y_t^n$. This will be dealt with in the next chapter.

For the nominal interest rate we simply assume a Taylor-style rule. But, not having fully ascertained whether we have one or two independent real variables in the system, we cannot make any further headway here.

# Chapter 7

# Steady State and Natural Levels

## 7.1 Introduction

In conventional DSGE models, we distinguish between steady states, often further specified as zero inflation ones, and natural levels of certain variables, specifically output, but [Christiano et al., 2010] also uses his model to examine natural rates of unemployment. In the previous chapter, we allowed ourself to rewrite the NKPC and Euler equations as deviations around the natural rates of employment, with the caveat that these values might not be well-defined. Specifically, the real per capita bond income of the optimising segment, which influences the optimal number of optimising households, is of concern. In this chapter, these issues will be addressed.

## 7.2 Consumption levels

In steady state, we will have $\frac{W_t^1 H_t^1}{W_t^0 H_t^0} = \frac{\alpha_1}{\alpha_0}$ from the optimality conditions of the (in steady state) representative intermediate goods producer. This means that the labour market income of the optimising households will be larger than that of the rule of thumb households, due to their greater importance in the production process. If we insert the budget constraints of each part of the family, we get:

$$\frac{H_t^1 P_t C_t^1 + H_t^1 Q_t B_t - B_{t-1} H_{t-1}}{P_t C_t^0 (1 - H_t^1)} = \frac{\alpha_1}{\alpha_0}$$

There is one specific point we shall make here. In [Galí et al., 2007], it is assumed steady state consumption levels for the two segments are equal, that is $C_t^0 = C_t^1$. This is done if necessary by the imposition of a targeted lump sum tax. Unfortunately in our model, this is not possible: Since households choose by their own volition to put in the effort to join the optimising segment, they will expect a return on their investment of effort. If $C_t^1$ were the same as $C_t^0$, for positive $F_k$, no households would choose to become optimising.

Another problem is that a differential lump sum tax in our model is not non-discretionary: Households can change their behaviour to avoid a differential tax of this type, by changing their effort levels to avoid the tax. Thus, we see that $C_t^1$ must be higher than $C_t^0$. This is also why we have assumed $\alpha_1 > \alpha_0$: The rationale for this choice is that it will push up wage levels for the optimising segment, increasing the scope for $C_t^1$ to remain higher than $C_t^0$.

Being that $C_t^1$ and $C_t^0$ are generally not equal, we might consider the related question as to what value $\frac{C_t^0}{Y_t}$ take in steady state and under price flexibility. Recall that by equilibrium, and and using the internal constraints of the family, $C_t^0$ and $Y_t$ both divide cleanly by $C_t^1$. Thus:

$$\frac{C_t^0}{Y_t} = \frac{F_{crt}(H_t^1, H_t^0)}{F_{agr}(H_t^1, H_t^0)}$$

Immediately we see that the ratio is exclusively determined by $H_t^1$ and $H_t^0$, and independent of technology, interest rates and the like.[1] Thus, if $H_t^1$ and $H_t^0$ have well-defined levels under price flexibility, then this ratio also takes on a well-defined value. The same applies analogously for $C_t^1$. Thus, let us look at employment levels.

## 7.3 Employment levels

The natural levels of employment will be determined by equilibrium when price frictions are zero. Given a path for the natural level of output, and using the optimality conditions of the family to tie $H_t^1$ and $H_t^0$ together, along with the first order conditions of the producers, it seems plausible that natural levels of employment have meaningful interpretations in our model. But recall the discussion in the introduction regarding real per capita bond income, which strongly indicates we need a path for bond holdings as well,

[1] Since neither of the two functions are dependent on these things.

which in turn will be influenced by the Euler equation. Time constraints resulting from the late stage at which this issue was noticed mean that this has not been studied further.

Steady state employment levels in our model provide an interesting topic for analysis as well. Refer to the intratemporal equilibrium condition of the family under simplified utility, and set $H_t^1 = H_{t-1}^1 = H^1$. Additionally, $P_t$ can be normalised to one, since there is zero inflation, and bond holdings as well as consumption must be constant. In this case, we get:

$$C^1 \left(H^1\right)^{2+\varphi} = BH^1$$

which implies

$$\frac{B}{C^1} = H^{1+\varphi}.$$

Thus given a steady state $B$ for the optimising family, $H^0$ will be tied down using

$$H^0 = S\left(\frac{B}{H^{1+\varphi}}, H^1\right).$$

Note though, that $\frac{\partial H^0}{\partial B} > 0$. This can be related again to the fact that the larger income available to optimising households motivates households to join the labour market. Indeed, $\frac{\partial H^1}{\partial B} > 0$ as well. This state-dependency in the family dynamics was certainly not expected at the outset, but it allows us to regard the games with a different lens.

# Chapter 8

# Calibration

## 8.1 Introduction

When choosing parameter values for any model, we seek values that are both justifiable empirically, and conductive to the mathematical assumptions of the model. In the model at hand, we will pay close heed to this second objective, since we have several probabilities and variables such as the employment levels which are restricted to certain intervals.

## 8.2 The family dynamics

The family dynamics, as mentioned before, are an expanded version of the model found in [Christiano et al., 2010]. As such, certain parameters in our model will be chosen similarly.[1] Recall that we have previously set $\sigma_j = \sigma_k = 1$ to allow analytic solutions to the integrals. The parameter values we choose are reported in Table 8.1.

We can test these values within the family model by trying reasonable values for $m_t^{k,1}, m_t^{j,h}$ and $\log c_t(0,0)$. If we let $m_t^{k,1} = \frac{1}{2}$, $m_t^{j,h} = \frac{3}{4}$, and $\log c_t(0,0)$ be free, we ensure all households have optimal success probabilities less than one in both games, and employment levels of $H_t^1 = 0.121$, $H_t^0 = 0.406$ for example.

Note that we have chosen $\eta$ and $a_e$ to be half that of what Christiano has. The reason for this is partially to compensate for our choice of $\sigma_j = 1$, and partially due to the fact that households with medium credit aversion in our model will have a greater incentive to join the labour market than households

[1] Christiano et al embed their model in various DSGE models, and occasionally choose very different parameter values dependent on the rest of the model.

| Variable name | Our value | Value in Christiano |
|---|---|---|
| $\eta$ | 0.43 | 0.86 |
| $a_e$ | 0.265 | 0.53 |
| $k_1$ | 0.4 + shock | - |
| $(b_j)$ | | - |
| $(b_g)$ | | - |
| $(b_x)$ | | - |
| $b_e$ | 0.2 | - |
| $F_j$ | 1.39 | 1.39 |
| $F_k$ | 3.8 | - |
| $\varsigma_j$ | 4.64 | 4.64 |
| $\varsigma_k$ | 2 | - |

Table 8.1: The parameters of the family dynamics

in Christiano do, since having a job in our model is a precondition for getting credit.

### 8.2.1 Maximal value of the marginals

Of course, our marginal levels will be determined by the macro-economic equilibrium, and we will need to check, especially for high employment levels, that the corresponding marginal levels and probabilities are admissible. Indeed, with the chosen parameter values, we have maximal permissible values for the two marginals given by the condition that the household with the least aversion to both work and credit chooses effort levels such that optimal probabilities are less than one. The choice of $e_t^{k,1}$ turns out not to be binding, so we have:

$$1 \geq p(e_t^{j,m})_{\phi=0,\psi=0} = 0.430 + 0.112 m_t^{k,1} + 0.651 m_t^{j,h} + 0.022 \left(m_t^{k,1}\right)^2 . \quad (8.1)$$

### 8.2.2 The Taylor-approximations

With the chosen parameter values, the elements of the matrix $J$ become:

$$
\begin{aligned}
J[1,1] =& 0.021 \left(m_t^{k,1}\right)^2 + 0.172 m_t^{j,h} + 0.005 \left(m_t^{k,1}\right)^3 \\
&+ 0.052 \left(m_t^{j,h}\right)^2 m_t^{k,1} + 0.114 m_t^{j,h} m_t^{k,1} + 0.130 \left(m_t^{j,h}\right)^2 \\
&+ 0.027 m_t^{j,h} \left(m_t^{k,1}\right)^2 + 0.030 m_t^{j,h} + 0.004 m_t^{j,h} \left(m_t^{k,1}\right)^3 \\
J[1,2] =& 0.057 \left(m_t^{k,1}\right)^2 + 0.009 \left(m_t^{k,1}\right)^3 + 0.261 m_t^{j,h} m_t^{k,1} \\
&+ 0.052 m_t^{j,h} \left(m_t^{k,1}\right)^2 + 0.172 m_t^{k,1} \\
J[2,1] =& 0.003 \left(m_t^{k,1}\right)^2 - 0.172 m_t^{j,h} - 0.002 \left(m_t^{k,1}\right)^3 \\
&- 0.052 \left(m_t^{j,h}\right)^2 * m_t^{k,1} - 0.001 m_t^{j,h} m_t^{k,1} - 0.130 \left(m_t^{j,h}\right)^2 \\
&- 0.004 m_t^{j,h} \left(m_t^{k,1}\right)^2 + 0.044 m_t^{k,1} - 0.004 m_t^{j,h} \left(m_t^{k,1}\right)^3 \\
J[2,2] =& - 0.001 \left(m_t^{k,1}\right)^3 - (0.430 + (0.652 m_t^{j,h})(m_t^{k,1} - 1) \\
&+ 0.391 m_t^{j,h} m_t^{k,1} - 0.052 m_t^{j,h} \left(m_t^{k,1}\right)^2 + 0.258 m_t^{k,1}
\end{aligned}
$$

where a few higher-order terms with coefficients less than 0.001 have been omitted. The inverse matrix of this, taken in a particular point, will then be used to create the Taylor approximations using the employment levels.

A sensible choice for this inversion point might be what we might term balanced maximal participation, namely a point $m_t^{k,1} = m_t^{j,h}$ such that (8.1) is satisfied with equality. Solving this quadratic equation gives $m_t^{k,1} = m_t^{j,h} = 0.731$ truncated to three significant figures. In this case, the matrices $J_H$ and $J_h$, along with their respective inverses, become:

$$
\begin{aligned}
J_H &= \begin{bmatrix} 0.324 & 0.320 \\ -0.186 & 0.620 \end{bmatrix}, & J_h &= \begin{bmatrix} 1.737 & 1.714 \\ -0.532 & 1.774 \end{bmatrix} \\
J_H^{-1} &= \begin{bmatrix} 2.383 & -1.229 \\ 0.715 & 1.245 \end{bmatrix}, & J_h^{-1} &= \begin{bmatrix} 0.444 & -0.429 \\ 0.133 & 0.435 \end{bmatrix}
\end{aligned}
$$

and using that $H_t^1(0.731, 0.731) = 0.186$ and $H_t^0(0.731, 0.731) = 0.350$, we can write out the Taylor approximations to the marginals as:

$$m_t^{k,1} = 0.716 + 2.383H_t^1 - 1.229H_t^0$$
$$m_t^{j,h} = 0.162 + 0.715H_t^1 + 1.245H_t^0$$

along with

$$m_t^{k,1} = 1.026 + 0.444h_t^1 - 0.429h_t^0$$
$$m_t^{j,h} = 1.412 + 0.133h_t^1 + 0.435h_t^0.$$

Note of course that dependent on the chosen inversion point, these coefficients might change sign. Indeed, recall that our inversion point is in the upper permissible range for our model.

One thing that is worth noting in our model is that, likely due to the choice of $\sigma_j = 1$, the total number of households in employment is rather low, only slightly above 50%, even at this upper permissible level for the marginals and high level of $k_1$. This is definitely a major issue when interpreting the results obtained from this model. It also means that we must ensure a calibration for the rest of the model that is compatible with these low employment levels, even during upturns, to avoid exceeding the permissible ranges.

### 8.2.3 The impact of $k_1$

Note in the above that we have silently folded $k_1$ into the coefficients. This will be rectified now. In Tables 8.2 and 8.3, we report the coefficients of the Taylor approximations as $k_1$ is varied. In each case, we leave all other parameters the same and keep the inversion point as $m_t^{k,1} = m_t^{j,h} = 0.731$. Note that for the value $k_1 = 0.5$ reported in the table, we exceed the permissible range for the probabilities in the inversion point, but decreasing $k_1$ does not pose these problems.

We can use this table to compare the optimality conditions of the family as $k_1$ changes, and to gain an understanding of the maximal permissible employment levels as we change $k_1$. Recall that the implied $m_t^{k,1}$ and $m_t^{j,h}$ should be below 1 for the values of $H_t^1$ and $H_t^0$ considered. This will be explored further in the next chapter.

## 8.3 The aggregate variables

In addition to the $\mu$ and $\kappa$ parameters resulting from the internal dynamics of the family, we need to choose suitable values for the other variables in the

| $k_1$ | $\kappa_k$ | $\kappa_{k1}$ | $\kappa_{k0}$ | $\kappa_j$ | $\kappa_{j1}$ | $\kappa_{j0}$ |
|---|---|---|---|---|---|---|
| 0.5 | 0.713 | 1.742 | -1.238 | 0.156 | 0.744 | 1.278 |
| 0.4 | 0.716 | 2.383 | -1.229 | 0.162 | 0.715 | 1.245 |
| 0.3 | 0.719 | 3.342 | -1.207 | 0.169 | 0.691 | 1.211 |
| 0.2 | 0.722 | 4.922 | -1.160 | 0.175 | 0.677 | 1.177 |
| 0.1 | 0.726 | 7.988 | -1.054 | 0.182 | 0.689 | 1.144 |
| 0.0 | 0.729 | 16.40 | -0.735 | 0.189 | 0.797 | 1.113 |

Table 8.2: The effects of varying $k_1$ on the coefficients of the first Taylor approximations.

| $k_1$ | $\mu_k$ | $\mu_{k1}$ | $\mu_{k0}$ | $\mu_j$ | $\mu_{j1}$ | $\mu_{j0}$ |
|---|---|---|---|---|---|---|
| 0.5 | 0.872 | 0.406 | -0.389 | 1.448 | 0.174 | 0.401 |
| 0.4 | 1.026 | 0.444 | -0.429 | 1.412 | 0.133 | 0.435 |
| 0.3 | 1.213 | 0.474 | -0.462 | 1.368 | 0.098 | 0.463 |
| 0.2 | 1.437 | 0.489 | -0.481 | 1.316 | 0.067 | 0.488 |
| 0.1 | 1.688 | 0.473 | -0.468 | 1.259 | 0.041 | 0.508 |
| 0.0 | 1.812 | 0.348 | -0.346 | 1.191 | 0.017 | 0.524 |

Table 8.3: The effects of varying $k_1$ on the coefficients of the second Taylor approximations.

| Variable name | Value | Comment |
|---|---|---|
| $\beta$ | 0.99 | Discount factor. |
| $\theta$ | 0.75 | Price stickiness. |
| $\varphi$ | 1 | Simplified utility function CRRA parameter. |
| $\alpha_1$ | 0.5 | Importance of $H_t^1$ to production process. |
| $\alpha_0$ | 0.25 | Importance of $H_t^0$ to production process. |
| $\varepsilon$ | 6 | Consumption index parameter. |

Table 8.4: The remaining parameters of the DSGE model

model. Note that the chosen $\beta$ is often seen when the period is assumed to be three months, $\varepsilon$ is chosen to correspond to $\lambda_f$ in Christiano, $\alpha_1$ and $\alpha_0$ are chosen so as to sum to less than one and $\varphi$ would ideally be calibrated to approximate the micro-founded $Z(H_t^1, H_t^0)$ function, but has here just been chosen to reflect quadratic disutility, in correlation with the households' effort cost.

# Chapter 9

# The effect of a shock to $k_1$

## 9.1 Introduction

One of the main motivations for constructing any economic model is to examine the effects of changes, both to variables and parameters. In the case of a DSGE model, it is common practice to examine the effects of parameter shocks to a model in steady state.

In this current thesis, we face a few difficulties with this approach. First of all, the difficulty in finding an expression for steady state, and the deviations of variables away from steady state. Secondly, and linked to this, the time to implement the model in Dynare was too large. Thus, in this chapter we shall attempt to trace the effects of a shock to $k_1$ "by hand" as it were. Let us begin with the simplified utility function, which allows easy analysis of the optimality conditions.

## 9.2 Effects on incentive compatibility

Assume the output gap is currently zero. This implies $b_x x_{t-1} = 0$. Thus the remaining components of $k_1$ are $b_g g_t$ and $b_j j_t$. Suppose an increase in $g_t$ is implemented, which was the government pressure on banks to extend credit.

In the internal model of the family, $m_t^{j,h}$ is not affected. Neither is $m_t^{k,1}$. But $m_t^{j,m}$ is: its slope changes. That is more households with medium credit aversion will join the job market. This means that $H_t^1$ and $H_t^0$ will both increase: some will succeed, and some will fail, but more households in total are joining the job market game and credit market game.

In the internal model we cannot predict what effect this will have on the consumption levels though. The ratios will not change, since these are determined by $m_t^{k,1}$ and $m_t^{j,h}$, but the levels might change. Note though that this effect thus functions to put upward pressure on $C_t^0$: Less unemployed households to support and more households to support them will bring up the average consumption. One problem with this analysis is that it sidesteps the fact of whether consumption levels will change within the family. This is easier to consider from the DSGE model. Note though that increasing government pressure on banks to lend out money does not directly affect $m_t^{k,1}$: people that had no intention of joining the credit market due to high credit aversion parameters will not join even if it is easier to join. To achieve this effect, we would need consumption levels promised to the households to change.

In the external model meanwhile, we see that $k_1$ enters through the parameters of the inverse function to the marginals. From Table 8.2, we can see that an increase in $k_1$, will have different effects depending on its initial level, along with the level of $H_t^1$ and $H_t^0$ respectively. The rationale for this is clear: the level of $m_t^{j,h}$ and $m_t^{k,1}$ that brought about a particular $H_t^1$ and $H_t^0$ are going to be affected by the shape of the derived $m_t^{j,m}$ function.

To progress, let us assume that $k_1 = 0.3$, and that $H_t^1 = 0.1$, $H_t^0 = 0.4$. Assume $\alpha_1 = \frac{1}{2}$ and $\alpha_0 = \frac{1}{4}$. Thus, by the first order condition of the aggregate production function, we can set $W_t^1 = 8$ and $W_t^0 = 1$. Assume $P_t = 1$, and thus

$$C_t^0 = \frac{W_t^0 H_t^0}{1 - H_t^1} = \frac{4}{9}.$$

By incentive compatibility, we can thus solve for the implied $C_t^1$, using the parameter values established previously and inserting into (4.1):

$$\begin{aligned}
\frac{4}{9} =& C_t^1 \left(e^{-F_k - 2\varsigma_k \mu_k}\right) (H_t^1)^{-2\varsigma_k \mu_{k1}} (H_t^0)^{-2\varsigma_k \mu_{k0}} \\
& \times \left[ \frac{H_t^0}{1 - H_t^1} + \left(1 - \frac{H_t^0}{1 - H_t^1}\right) \left(e^{-F_j - 2\varsigma_j \mu_j}\right) (H_t^1)^{-2\varsigma_j \mu_{j1}} (H_t^0)^{-2\varsigma_j \mu_{j0}} \right] \\
=& C_t^1 \left(e^{-3.8 - 2 \cdot 2 \cdot 1.213}\right) (0.1)^{-2 \cdot 2 \cdot 0.474} (0.4)^{2 \cdot 2 \cdot 0.462} \\
& \times \left[ \frac{0.4}{1 - 0.1} + \left(1 - \frac{0.4}{1 - 0.1}\right) \left(e^{-1.39 - 2 \cdot 4.64 \cdot 1.368}\right) (0.1)^{-2 \cdot 4.64 \cdot 0.098} (0.4)^{-2 \cdot 4.64 \cdot 0.463} \right] \\
=& 0.00112 \cdot C_t^1,
\end{aligned}$$

which implies that the $C_t^1$ required by incentive compatibility in this case is $C_t^1 \approx 397$.

It is worth commenting on this very large difference between the two values: Households are assumed to have logarithmic utility of consumption, but their cost of credit maintenance and effort costs are either linear or quadratic. Thus, if we compare utilities stemming from consumption, we see that the difference between the two numbers is much smaller:

$$\log(397) - \log\left(\frac{4}{9}\right) = 2.95.$$

Recall that households choosing to participate in the credit market game do so only if their ex-ante utility of joining the game is positive. In this sense, the magnitude of this number matches closely with our fixed cost of credit, $F_k = 3.8$. Initially, this seems to suggest that our inverse function is underestimating the required $C_t^1$, since even with a very small effort and very low credit aversion, the difference in realised utility in the success state is negative. But upon further thought, note that we are comparing $C_t^0$ and $C_t^1$, and even though $C_t^1 = c_t(1,1)$, then $c_t(1,0)$, which is the relevant "failure" state for a household participating in the credit game, has the property $c_t(1,0) > C_t^0$.

What happens when $k_1$ increases, either as a result of government pressure, or a positive output gap? If we set $k_1 = 0.4$, keep $H_t^1$ and $H_t^0$ unchanged, and redo the calculation for the implied $C_t^1$ using the $\mu$ values from the previous chapter, we get $C_t^0 = 0.00203C_t^1$, and thus $C_t^1 = 218.9$. This is interesting, since this effect is independent of our assumption that one segment of the family should be rule of thumb, but may act to further exaggerate the pro-cyclical nature of rule of thumb consumption. The lower implied $C_t^1$ is exclusively a result of increasing the chance of success in the credit market game, and thus to obtain the same ex-ante utility, a lesser outcome in the success state is required.

One problem with this analysis of course is we ignored changes in employment levels. Indeed, as argued previously, a higher $k_1$ should increase both $H_t^1$ and $H_t^0$. So if $k_1$ increases, we should expect the employment levels in the new equilibrium to adjust as well.

## 9.3 Impacts on aggregate production

Changes to $k_1$ do not directly affect producers, in that neither $A, \alpha_1, \alpha_0$ nor $\varepsilon$ are affected. But note that if we use the micro-founded utility function, then optimal price setting will be affected. Specifically, $Q_t$, defined from the Euler equation, will be affected by the changes in $Z(H_t^1, H_t^0)$.

But even though producers are not affected directly, we see that a significant effect of $k_1$ in the dynamic system is through the NKPC. Referring to the expression for $mc_t$ that (6.3) is based on, we see that marginal cost in our model is a function of employment levels and also acting through $F_{crt}$, which we have noted above changes when $k_1$ changes. Additionally, the Euler equation can also be expressed using $F_{crt}$, so this will also be affected by a shift in $k_1$.

## 9.4 The optimality conditions of the family

Recall the dependence of the optimising segment's budget constraint on per capita bond payouts. This implies that our intratemporal optimality condition, (4.4) is state dependent. Under the simplified utility specification, we have:

$$C_t^1 \left(H_t^1\right)^2 + \varphi = \frac{B_{t-1} H_{t-1}^1}{P_t}$$

For shifts in $k_1$, note that the RHS is not affected. Thus, the LHS must stay the same. As previously noted, if $k_1$ increases and employment levels are held fixed, then $C_t^1$ must decrease by incentive compatibility.[1] But we see that this is not compatible with the optimality condition. Indeed, for $\varphi > -2$, $C_t^1$ and $H_t^1$ must move in opposite directions for the optimality conditions of the family to be maintained.

If we assume that $C_t^1$ does indeed decrease and $H_t^1$ increases as a result of the shift in $k_1$, we can maintain the intratemporal optimality of the family. Turning now to intertemporal optimality, if the increase in $k_1$ is assumed temporary, then we see there are conflicting effects regarding our expectation on $C_{t+1}^1$: on the one hand, the greater number of optimising households, and their smaller consumption this period may lead to increased bond purchases. This means the income available to optimising households next period is higher, which may translate into increased consumption. On the other hand, if per capita income for the optimising family falls along with $C_t^1$, since $H_t^1$ increases, lowering the wage rate, then per capita bond purchases might be stagnant or fall. In this case $C_{t+1}^1$ may decrease further or revert to its previous value.

## 9.5 Effects on aggregate production

Note though that by the above optimality condition for the family, for $\varphi > -1$, then $C_t^1 H_t^1$ is going to increase when $k_1$ increases. This can be verified

[1] Assuming the wage levels do not change, then income for rule of thumb households remains unchanged due to the change in $k_1$.

as follows:

Denote $C_a^1$ and $H_a^1$ as the values with $k_1 = 0.3$. Denote by $C_b^1$ and $H_b^1$ the values with $k_1 = 0.4$. In both cases, the intratemporal optimality for the family must be satisfied. Assume $\varphi = 1$. Thus:

$$\frac{C_a^1}{C_b^1} = \left(\frac{H_b^1}{H_a^1}\right)^3$$

Note that for a decrease in consumption, meaning that the fraction on the left is less than one, this implies $\frac{H_b^1}{H_a^1} > \frac{C_a^1}{C_b^1}$. Thus, we get as postulated that $C_b^1 H_b^1 > C_a^1 H_a^1$.

This is interesting, since it implies, by goods market clearing, that at least one component of aggregate production, which was defined as $Y_t = C_t^1 H_t^1 + (1 - H_t^1)C_t^0$, increases. The second component, consumption for rule of thumb households, is less obvious, but if we assume $H_t^0$ increases, along with the fact that $H_t^1$ also increasing, then we see, by the internal budget constraint of the rule of thumb family, that $C_t^0$ must also increase.[2] Thus, there is a preliminary indication that increases in $k_1$ increase aggregate production.

## 9.6 Impacts on inflation

It is worth recalling that our model is standard enough that inflation is determined by expectations regarding future inflation, along with deviations in marginal cost. Marginal costs are also, since we assumed DRS, high when aggregate production is high. As noted above, an increase in $k_1$ might bring about an increase in aggregate production. And thus, we might see an increase in inflation.

[2]But this would in turn increase the implied $C_t^1$ required by incentive compatibility!

# Chapter 10

# Conclusion

In this thesis, a relatively large game-theoretical model of household behaviour has been embedded into a DSGE model. Although time constraints precluded sophisticated numerical simulation in Dynare, and little attention has been paid to aspects such as the exogenous processes of shocks, the exercise has still been fruitful, highlighting several curious mechanisms in the model and giving tentative support to the hypothesis that government pressure on banks to increase credit availability can affect aggregate output significantly.

## 10.1 Main results

In this section, three main results will be highlighted. Firstly, the state dependency of the family's optimisation problem. Due to the variable size of the two segments from one period to the next, we had to be very careful regarding a consistent specification of the budget constraints. The need to express variables at their per capita levels meant that past behaviour of the optimising segment would affect current optimal allocations by the family. This effect was unexpected, and complicated the first order conditions of the family significantly; to the extent that several standard substitutions to obtain closed form solutions to the DSGE system were not possible. But it highlighted an interesting aspect of endogenous family composition where the optimising behaviour of households within the family needs to be very closely considered.

Secondly, the family construction in which households must invest effort to transition from one stage to another provided a novel connection between the behaviour of rule of thumb households and optimising households. Incentive compatibility relations within the family imposed alternative assumptions on consumption levels in steady state and would complicate the imposition of taxes to fund a government, being that lump sum taxes would be avoidable. Also, the game-theoretical approach to household behaviour allowed us to characterise several distinct strategies and provided a detailed framework for understanding heterogeneity among households.

Finally, we made some headway on answering the main research question of the thesis, by examining the effect of changes to $k_1$, which encompassed government pressure on banks, on employment levels, consumption levels, and aggregate output. We saw that $k_1$ affected the model in several ways, and tentatively concluded that government pressure on banks to increase lending may be warranted as an effective measure to raise aggregate output.

## 10.2 Limitations

The main limitation of the theoretical framework for this thesis is arguably our modelling of bank behaviour and government. Being that the thesis set out to investigate government pressure on banks, it would certainly have been preferable to include an endogenous model of government and pressure on banks, either through a nationalisation decision, or through a mechanism by which the government must choose between changes to expenditure and applying pressure on the banks, reflecting the limited number of policy changes that can be made during any period. Also, a consistent model of the banks would certainly have been beneficial, perhaps by introducing them as a separate class of producer, owned by patient households, similar to how [Iacoviello and Neri, 2010] model a housing market.

Limitations in scope and power of our model were also imposed, distracting from a particularly interesting sub-question of the original research question. By removing the ability of unemployed households to gain credit, we were unable to extend the model to considerations regarding which households should be granted credit. Admittedly, allowing unemployed households to obtain credit would likely have required a reason for our family to choose a particular level of unemployed optimising households, but if we further introduced some kind of stickiness to the total size of the two segments, we would be able to discuss aspects such as how long to allow unemployed households to maintain credit, or in other words, how long to keep bad loans on the books. This aspect might even allow us to re-interpret the model to describe policy responses regarding sub-prime mortgages.

Finally, computational issues even in the reduced model presented significant problems with the analysis and conclusions. Chief among these is probably the necessity of invoking the simplified utility function for the family with insufficient justification and modification of household and family behaviour. It would have been preferable to obtain a separable polynomial approximation to the utility function, which would allow us to express the first-order conditions for the family more accurately, while retaining tractability.

## 10.3 Further research

Due to both time constraints and unexpected complications in the interactions between the game-theoretical model and the rest of the macroeconomic system, the scope of this thesis is smaller than originally envisioned. The obvious next step is to close the model fully, and implement it in Dynare to allow comparison with related models and a proper analysis of the dynamics resulting from shocks to $k_1$ in combination with technology shocks.

As mentioned above, the most pressing extension to the model would certainly be some sort of persistence in household states. One might imagine Calvo-style frictions on the number of optimising households, and combining this with a labour union to provide a model for wage levels. Additionally, if persistence in states were implemented, then the single family could be split into two, each separately optimising, since households which start the period in the optimising family would allow us to construct an aggregate behaviour for that family to ensure for example that as few households as possible are forced out of the credit markets.

Finally, introducing capital into the model would enable several aspects of inequality in wealth and income to be studied. In this way, we would get an interpretation of why households would be unemployed and optimising, if we specified the firms to be owned by the optimising households only. We could then imagine households with high aversion to work, but low credit aversion to be rentiers, choosing to accumulate and manage capital instead of working.

# Appendix A

# Obtaining the Employment Levels

## A.1 Basic components

As mentioned in the thesis, we can aggregate the optimal behaviour of the households to obtain the employment levels. This involves handling several integrals, which shall be dealt with in this appendix. Note that we remain as general as possible in this appendix, but also provide results for the reduced model. But first, when evaluating the integrals below, we shall need some basic relations, which will be established here. Note that the antiderivative of $f_j(m_t^{j,h}, \phi)$ is

$$\int f_j(m_t^{j,h}, \phi) d\phi = \varsigma_j \left( (1 + \sigma_j)(m_t^{j,h})^{\sigma_j} \phi - \phi^{1+\sigma_j} \right),$$

and furthermore that

$$\int_0^{m_t^{j,h}} f_j(m_t^{j,h}, \phi) d\phi = \varsigma_j \sigma_j (m_t^{j,h})^{1+\sigma_j}.$$

Also,

$$\begin{aligned}
&\int_0^{m_t^{j,h}} f_j^2(m_t^{j,h},\phi)d\phi \\
&= \int_0^{m_t^{j,h}} \varsigma_j^2(1+\sigma_j)^2 \left((m_t^{j,h})^{2\sigma_j} - 2(m_t^{j,h})^{\sigma_j}\phi^{\sigma_j} + \phi^{2\sigma_j}\right) d\phi \\
&= \varsigma_j^2(1+\sigma_j)^2 \left[(m_t^{j,h})^{2\sigma_j}\phi - \frac{2(m_t^{j,h})^{\sigma_j}\phi^{1+\sigma_j}}{1+\sigma_j} + \frac{\phi^{1+2\sigma_j}}{1+2\sigma_j}\right]_0^{m_t^{j,h}} \\
&= \varsigma_j^2(1+\sigma_j)^2 \left((m_t^{j,h})^{2\sigma_j} m_t^{j,h} - \frac{2(m_t^{j,h})^{\sigma_j}(m_t^{j,h})^{1+\sigma_j}}{1+\sigma_j} + \frac{(m_t^{j,h})^{1+2\sigma_j}}{1+2\sigma_j}\right) \\
&= \left(\frac{2\varsigma_j^2\sigma_j^2(1+\sigma_j)(m_t^{j,h})^{1+2\sigma_j}}{1+2\sigma_j}\right).
\end{aligned}$$

The above relations hold analogously for $f_k(m,\psi)$. Additionally, we will need several antiderivatives relating to $f_k(m,\psi)$.

$$\begin{aligned}
&\int_0^{m_t^{k,0}} f_k(m_t^{k,1},\psi)f_k(m_t^{k,0},\psi)d\psi \\
&= \int_0^{m_t^{k,0}} \varsigma_k^2(1+\sigma_k)^2 \left((m_t^{k,1})^{\sigma_k}(m_t^{k,0})^{\sigma_k} - (m_t^{k,0})^{\sigma_k}(\psi)^{\sigma_k} - (m_t^{k,1})^{\sigma_k}(\psi)^{\sigma_k} + (\psi)^{2\sigma_k}\right) d\psi \\
&=\varsigma_k^2(1+\sigma_k)^2 \left[(m_t^{k,1})^{\sigma_k}(m_t^{k,0})^{\sigma_k}\psi - \frac{(m_t^{k,0})^{\sigma_k}(\psi)^{1+\sigma_k}}{1+\sigma_k} - \frac{(m_t^{k,1})^{\sigma_k}(\psi)^{1+\sigma_k}}{1+\sigma_k} + \frac{(\psi)^{1+2\sigma_k}}{1+2\sigma_k}\right]_0^{m_t^{k,0}}
\end{aligned}$$

And also:

$$\begin{aligned}
&\int_0^{m_t^{k,0}} f_k^2(m_t^{k,1},\psi)f_k(m_t^{k,0},\psi)d\psi \\
&= \int_0^{m_t^{k,0}} \varsigma_k^3(1+\sigma_k)^3 \left((m_t^{k,1})^{2\sigma_k}(m_t^{k,0})^{\sigma_k} - 2(m_t^{k,1})^{2\sigma_k}\psi^{\sigma_k} - 2(m_t^{k,1})^{\sigma_k}(m_t^{k,0})^{\sigma_k}\psi^{\sigma_k}\right) d\psi \\
&\quad + \int_0^{m_t^{k,0}} \varsigma_k^3(1+\sigma_k)^3 \left(2(m_t^{k,1})^{\sigma_k}\psi^{2\sigma_k} + (m_t^{k,0})^{\sigma_k}\psi^{2\sigma_k} - \psi^{3\sigma_k}\right) d\psi \\
&=\varsigma_k^3(1+\sigma_k)^3 \left[(m_t^{k,1})^{2\sigma_k}(m_t^{k,0})^{\sigma_k}\psi - 2\frac{(m_t^{k,1})^{2\sigma_k}\psi^{1+\sigma_k}}{1+\sigma_k} - 2\frac{(m_t^{k,1})^{\sigma_k}(m_t^{k,0})^{\sigma_k}\psi^{1+\sigma_k}}{1+\sigma_k}\right]_0^{m_t^{k,0}} \\
&\quad + \varsigma_k^3(1+\sigma_k)^3 \left[2\frac{(m_t^{k,1})^{\sigma_k}\psi^{1+2\sigma_k}}{1+2\sigma_k} + \frac{(m_t^{k,0})^{\sigma_k}\psi^{1+2\sigma_k}}{1+2\sigma_k} - \frac{\psi^{1+3\sigma_k}}{1+3\sigma_k}\right]_0^{m_t^{k,0}}
\end{aligned}$$

For $f_k^3$, the antiderivative is:

$$
\begin{aligned}
&\int_{m_t^{k,0}}^{m_t^{k,1}} f_k^3(m_t^{k,1},\psi)d\psi\\
=&\int_{m_t^{k,0}}^{m_t^{k,1}} \varsigma_k^3(1+\sigma_k)^3\left((m_t^{k,1})^{3\sigma_k}-3(m_t^{k,1})^{2\sigma_k}\psi^{\sigma_k}+3(m_t^{k,1})^{\sigma_k}\psi^{2\sigma_k}-\psi^{3\sigma_k}\right)d\psi\\
=&\varsigma_k^3(1+\sigma_k)^3\left[(m_t^{k,1})^{3\sigma_k}\psi-\frac{3(m_t^{k,1})^{2\sigma_k}\psi^{1+\sigma_k}}{1+\sigma_k}+\frac{3(m_t^{k,1})^{\sigma_k}\psi^{1+2\sigma_k}}{1+2\sigma_k}-\frac{\psi^{1+3\sigma_k}}{1+3\sigma_k}\right]_{m_t^{k,0}}^{m_t^{k,1}}
\end{aligned}
$$

Also

$$
\begin{aligned}
&\int_{m_t^{k,0}}^{m_t^{k,1}} f_k^4(m_t^{k,1},\psi)d\psi\\
=&\int_{m_t^{k,0}}^{m_t^{k,1}} \varsigma_k^4(1+\sigma_k)^4\left((m_t^{k,1})^{4\sigma_k}-4(m_t^{k,1})^{3\sigma_k}\psi^{\sigma_k}+6(m_t^{k,1})^{2\sigma_k}\psi^{2\sigma_k}\right)d\psi\\
&+\int_{m_t^{k,0}}^{m_t^{k,1}} \varsigma_k^4(1+\sigma_k)^4\left(-4(m_t^{k,1})^{1\sigma_k}\psi^{3\sigma_k}+\psi^{4\sigma_k}\right)d\psi\\
=&\varsigma_k^4(1+\sigma_k)^4\left[(m_t^{k,1})^{4\sigma_k}\psi-\frac{4(m_t^{k,1})^{3\sigma_k}\psi^{1+\sigma_k}}{1+\sigma_k}+\frac{6(m_t^{k,1})^{2\sigma_k}\psi^{1+2\sigma_k}}{1+2\sigma_k}\right]_{m_t^{k,0}}^{m_t^{k,1}}\\
&+\varsigma_k^4(1+\sigma_k)^4\left[-\frac{4(m_t^{k,1})^{1\sigma_k}\psi^{1+3\sigma_k}}{1+3\sigma_k}+\frac{\psi^{1+4\sigma_k}}{1+4\sigma_k}\right]_{m_t^{k,0}}^{m_t^{k,1}}
\end{aligned}
$$

And also:

$$\begin{aligned}
&\int_{m_t^{k,0}}^{m_t^{k,1}} f_k^5(m_t^{k,1},\psi)d\psi \\
=&\int_{m_t^{k,0}}^{m_t^{k,1}} \varsigma_k^5(1+\sigma_k)^5\left((m_t^{k,1})^{5\sigma_k}-5(m_t^{k,1})^{4\sigma_k}\psi^{\sigma_k}+10(m_t^{k,1})^{3\sigma_k}\psi^{2\sigma_k}\right)d\psi \\
&+\int_{m_t^{k,0}}^{m_t^{k,1}} \varsigma_k^5(1+\sigma_k)^5\left(-10(m_t^{k,1})^{2\sigma_k}\psi^{3\sigma_k}+5(m_t^{k,1})^{1\sigma_k}\psi^{4\sigma_k}-(m_t^{k,1})^{0\sigma_k}\psi^{5\sigma_k}\right)d\psi \\
=&\varsigma_k^5(1+\sigma_k)^5\left[(m_t^{k,1})^{5\sigma_k}\psi-\frac{5(m_t^{k,1})^{4\sigma_k}\psi^{1+\sigma_k}}{1+\sigma_k}+\frac{10(m_t^{k,1})^{3\sigma_k}\psi^{1+2\sigma_k}}{1+2\sigma_k}\right]_{m_t^{k,0}}^{m_t^{k,1}} \\
&+\varsigma_k^5(1+\sigma_k)^5\left[-\frac{10(m_t^{k,1})^{2\sigma_k}\psi^{1+3\sigma_k}}{1+3\sigma_k}+\frac{5(m_t^{k,1})^{1\sigma_k}\psi^{1+4\sigma_k}}{1+4\sigma_k}-\frac{\psi^{1+5\sigma_k}}{1+5\sigma_k}\right]_{m_t^{k,0}}^{m_t^{k,1}}
\end{aligned}$$

And also:

$$\begin{aligned}
&\int_{m_t^{k,0}}^{m_t^{k,1}} f_k^8(m_t^{k,1},\psi)d\psi \\
=&\int_{m_t^{k,0}}^{m_t^{k,1}} \varsigma_k^8(1+\sigma_k)^8\left((m_t^{k,1})^{8\sigma_k}-8(m_t^{k,1})^{7\sigma_k}\psi^{\sigma_k}+28(m_t^{k,1})^{6\sigma_k}\psi^{2\sigma_k}\right)d\psi \\
&+\int_{m_t^{k,0}}^{m_t^{k,1}} \varsigma_k^8(1+\sigma_k)^8\left(-56(m_t^{k,1})^{5\sigma_k}\psi^{3\sigma_k}+70(m_t^{k,1})^{4\sigma_k}\psi^{4\sigma_k}-56(m_t^{k,1})^{3\sigma_k}\psi^{5\sigma_k}\right)d\psi \\
&+\int_{m_t^{k,0}}^{m_t^{k,1}} \varsigma_k^8(1+\sigma_k)^8\left(28(m_t^{k,1})^{2\sigma_k}\psi^{6\sigma_k}-8(m_t^{k,1})^{1\sigma_k}\psi^{7\sigma_k}+\psi^{8\sigma_k}\right)d\psi \\
=&\varsigma_k^8(1+\sigma_k)^8\left[(m_t^{k,1})^{8\sigma_k}\psi-\frac{8(m_t^{k,1})^{7\sigma_k}\psi^{1+\sigma_k}}{1+\sigma_k}+\frac{28(m_t^{k,1})^{6\sigma_k}\psi^{1+2\sigma_k}}{1+2\sigma_k}\right]_{m_t^{k,0}}^{m_t^{k,1}} \\
&+\varsigma_k^8(1+\sigma_k)^8\left[-\frac{56(m_t^{k,1})^{5\sigma_k}\psi^{1+3\sigma_k}}{1+3\sigma_k}+\frac{70(m_t^{k,1})^{4\sigma_k}\psi^{1+4\sigma_k}}{1+4\sigma_k}-\frac{56(m_t^{k,1})^{3\sigma_k}\psi^{1+5\sigma_k}}{1+5\sigma_k}\right]_{m_t^{k,0}}^{m_t^{k,1}} \\
&+\varsigma_k^8(1+\sigma_k)^8\left[\frac{28(m_t^{k,1})^{2\sigma_k}\psi^{1+6\sigma_k}}{1+6\sigma_k}-\frac{8(m_t^{k,1})^{1\sigma_k}\psi^{1+7\sigma_k}}{1+7\sigma_k}+\frac{\psi^{1+8\sigma_k}}{1+8\sigma_k}\right]_{m_t^{k,0}}^{m_t^{k,1}}
\end{aligned}$$

## A.2 Evaluating the integrals

### A.2.1 integral of $q\left(0, e_t^{k,0}\right)$

This integral appears twice, it relates to the probability of households with low credit aversion obtaining credit when not having a job. It is relatively simple to integrate, and has a pleasing structure:

$$\begin{aligned}
&\int_0^1 \int_0^{m_t^{k,0}} q\left(0, e_t^{k,0}\right) d\psi d\phi \\
=&\int_0^1 \int_0^{m_t^{k,0}} k_0 + b_e e_t^{k,0} d\psi d\phi \\
=&\int_0^1 \int_0^{m_t^{k,0}} \left(k_0 + b_e^2 f_k(m_t^{k,0}, \psi)\right) d\psi d\phi \\
=&k_0 m_t^{k,0} + b_e^2 \varsigma_k \sigma_k \left(m_t^{k,0}\right)^{1+\sigma_k}
\end{aligned}$$

With $k_0 = (b_g g_t(0) - b_x x_{t-1})$.

### A.2.2 integral of $p\left(e_t^{j,h}\right)$

This integral appears twice. It relates to the probability of households with high credit aversion obtaining a job. It also has a rather pleasing structure, and appears in [Christiano et al., 2010].

$$\begin{aligned}
&\int_0^{m_t^{j,h}} \int_{m_t^{k,1}}^1 p\left(e_t^{j,h}\right) d\psi d\phi \\
=&\int_{m_t^{k,1}}^1 \int_0^{m_t^{j,h}} \eta + a_e^2 f_j(m_t^{j,h}, \phi) d\phi d\psi \\
=&\int_{m_t^{k,1}}^1 m_t^{j,h}\eta + a_e^2 \varsigma_j \sigma_j \left(m_t^{j,h}\right)^{1+\sigma_j} d\psi \\
=&\left(1 - m_t^{k,1}\right)\left(m_t^{j,h}\eta + a_e^2 \varsigma_j \sigma_j \left(m_t^{j,h}\right)^{1+\sigma_j}\right)
\end{aligned}$$

### A.2.3 integral of $p\left(e_t^{j,m}\right)$

This integral appears thrice, and relates to the probability of households with medium credit aversion obtaining a job. This is no longer quite as easy as above to evaluate:

$$\begin{aligned}&\int_0^{m_t^{j,m}}\int_{m_t^{k,0}}^{m_t^{k,1}} p\left(e_t^{j,m}\right) d\psi d\phi \\ =&\int_0^{m_t^{j,m}}\int_{m_t^{k,0}}^{m_t^{k,1}} \eta + a_e^2 f_j(m_t^{j,m},\phi) d\psi d\phi\end{aligned}$$

Since $m_t^{j,m}$ is a function of $\psi$, we must integrate with respect to $\phi$ first. Thus we obtain:

$$=\int_{m_t^{k,0}}^{m_t^{k,1}} \eta m_t^{j,m} + a_e^2 \varsigma_j \sigma_j (m_t^{j,m})^{1+\sigma_j} d\psi$$

For general $\sigma_j$, all we can hope to do is a rough numerical integration:

$$\begin{aligned}&\int_0^{m_t^{j,m}}\int_{m_t^{k,0}}^{m_t^{k,1}} p\left(e_t^{j,m}\right) d\psi d\phi \\ \approx&\left(m_t^{k,1}-m_t^{k,0}\right)\frac{\eta}{2}(m_t^{j,m}|_{m_t^{k,1}} + m_t^{j,m}|_{m_t^{k,0}}) \\ &+\left(m_t^{k,1}-m_t^{k,0}\right)\frac{a_e^2\varsigma_j\sigma_j}{2}\left(\left(m_t^{j,m}|_{m_t^{k,1}}\right)^{1+\sigma_j}+\left(m_t^{j,m}|_{m_t^{k,0}}\right)^{1+\sigma_j}\right)\end{aligned}$$

where the dropped values indicate evaluation in a point.

If $\sigma_j = 1$, we can find the exact result as follows:

$$\begin{aligned}&\int_{m_t^{k,0}}^{m_t^{k,1}} \eta m_t^{j,m} + a_e^2 \varsigma_j (m_t^{j,m})^2 d\psi \\ =&\frac{\eta}{2\varsigma_j}\int_{m_t^{k,0}}^{m_t^{k,1}} \varsigma_j 2 m_t^{j,m} d\psi + \frac{a_e^2}{4\varsigma_j}\int_{m_t^{k,0}}^{m_t^{k,1}} \varsigma_j^2 4 (m_t^{j,m})^2 d\psi\end{aligned}$$

This is useful because the contents of each of these integrals is a sum of the building blocks defined above (by using the definition of $m_t^{j,m}$ laid out in the text). Indeed, let us examine first:

$$\begin{aligned}
&\int_{m_t^{k,0}}^{m_t^{k,1}} \varsigma_j 2m_t^{j,m} d\psi \\
=&\int_{m_t^{k,0}}^{m_t^{k,1}} k_1 f_k\left(m_t^{k,1},\psi\right) + \frac{1}{2}b_e^2 f_k^2\left(m_t^{k,1},\psi\right) + \varsigma_j 2m_t^{j,h} d\psi \\
=&\varsigma_j 2m_t^{j,h}\int_{m_t^{k,0}}^{m_t^{k,1}} 1 d\psi \\
&+ k_1\int_{m_t^{k,0}}^{m_t^{k,1}} f_k\left(m_t^{k,1},\psi\right) d\psi \\
&+ \frac{1}{2}b_e^2\int_{m_t^{k,0}}^{m_t^{k,1}} f_k^2\left(m_t^{k,1},\psi\right) d\psi
\end{aligned}$$

All of the integrals in this final expression have been calculated as building blocks previously.

And then the more intricate integral given by:

$$\begin{aligned}
&\int_{m_t^{k,0}}^{m_t^{k,1}} \varsigma_j^2 4(m_t^{j,m})^2 d\psi \\
=&\int_{m_t^{k,0}}^{m_t^{k,1}} \left(k_1 f_k\left(m_t^{k,1},\psi\right) + \frac{1}{2}b_e^2 f_k^2\left(m_t^{k,1},\psi\right) + \varsigma_j 2m_t^{j,h}\right)^2 d\psi
\end{aligned}$$

expanding the square, we get:

$$\begin{aligned}
=&\int_{m_t^{k,0}}^{m_t^{k,1}} k_1^2 f_k^2\left(m_t^{k,1},\psi\right) + \frac{1}{4}b_e^4 f_k^4\left(m_t^{k,1},\psi\right) \\
&+ \varsigma_j^2 4(m_t^{j,h})^2 + 2\frac{b_e^2}{2}\varsigma_j 2m_t^{j,h} f_k^2\left(m_t^{k,1},\psi\right) \\
&+ 2k_1\frac{b_e^2}{2} f_k^3\left(m_t^{k,1},\psi\right) + 2k_1\varsigma_j 2m_t^{j,h} f_k\left(m_t^{k,1},\psi\right) d\psi \\
=&\varsigma_j^2 4(m_t^{j,h})^2\int_{m_t^{k,0}}^{m_t^{k,1}} 1 d\psi + 4k_1\varsigma_j m_t^{j,h}\int_{m_t^{k,0}}^{m_t^{k,1}} f_k\left(m_t^{k,1},\psi\right) d\psi \\
&+ \left(k_1^2 + 2b_e^2\varsigma_j m_t^{j,h}\right)\int_{m_t^{k,0}}^{m_t^{k,1}} f_k^2\left(m_t^{k,1},\psi\right) d\psi + k_1 b_e^2\int_{m_t^{k,0}}^{m_t^{k,1}} f_k^3\left(m_t^{k,1},\psi\right) d\psi \\
&+ \frac{1}{4}b_e^4\int_{m_t^{k,0}}^{m_t^{k,1}} f_k^4\left(m_t^{k,1},\psi\right) d\psi
\end{aligned}$$

Where, again, all of these integrals have been evaluated analytically previously.

Collecting terms, we get that in total:

$$\begin{aligned}
&\int_{m_t^{k,0}}^{m_t^{k,1}} \eta m_t^{j,m} + a_e^2 \varsigma_j (m_t^{j,m})^2 d\psi \\
&= \left(\eta m_t^{j,h} + a_e^2 \varsigma_j (m_t^{j,h})^2\right) \int_{m_t^{k,0}}^{m_t^{k,1}} 1 d\psi \\
&+ \frac{k_1}{2\varsigma_j} \left(\eta + 2a_e^2 \varsigma_j m_t^{j,h}\right) \int_{m_t^{k,0}}^{m_t^{k,1}} f_k\left(m_t^{k,1}, \psi\right) d\psi \\
&+ \left(\frac{a_e^2 k_1^2 + b_e^2 \left(\eta + 2a_e^2 \varsigma_j m_t^{j,h}\right)}{4\varsigma_j}\right) \int_{m_t^{k,0}}^{m_t^{k,1}} f_k^2\left(m_t^{k,1}, \psi\right) d\psi \\
&+ \frac{a_e^2 b_e^2 k_1}{4\varsigma_j} \int_{m_t^{k,0}}^{m_t^{k,1}} f_k^3\left(m_t^{k,1}, \psi\right) d\psi \\
&+ \frac{a_e^2 b_e^4}{16\varsigma_j} \int_{m_t^{k,0}}^{m_t^{k,1}} f_k^4\left(m_t^{k,1}, \psi\right) d\psi
\end{aligned}$$

### A.2.4 integral of $p\left(e_t^{j,l}\right)$

This integral also appears twice, and relates to the probability of households with low credit aversion obtaining a job.

$$\begin{aligned}
&\int_0^{m_t^{j,l}} \int_0^{m_t^{k,0}} p\left(e_t^{j,l}\right) d\psi d\phi \\
&= \int_0^{m_t^{j,l}} \int_0^{m_t^{k,0}} \eta + a_e^2 f_j(m_t^{j,l}, \phi) d\psi d\phi
\end{aligned}$$

Since $m_t^{j,l}$ is a function of $\psi$, we must integrate with respect to $\phi$ first.

$$\int_0^{m_t^{k,0}} \eta m_t^{j,l} + a_e^2 \varsigma_j \sigma_j (m_t^{j,l})^{1+\sigma_j} d\psi$$

As before, if $\sigma_j$ is general, then all we can do is a rough numerical integration:

$$\approx m_t^{k,0} \left(\frac{\eta}{2}(m_t^{j,l}|_0 + m_t^{j,l}|_{m_t^{k,0}}) + \frac{a_e^2 \varsigma_j \sigma_j}{2}\left(\left(m_t^{j,l}|_0\right)^{1+\sigma_j} + \left(m_t^{j,l}|_{m_t^{k,0}}\right)^{1+\sigma_j}\right)\right)$$

But if $\sigma_j = 1$, then again, we note that we can analytically integrate it by looking at the components of $m_t^{j,l}$.

That is:

$$\begin{aligned}
&\int_0^{m_t^{k,0}} \eta m_t^{j,l} + a_e^2 \varsigma_j \sigma_j (m_t^{j,l})^{1+\sigma_j} d\psi \\
=&\frac{\eta}{2\varsigma_j} \int_0^{m_t^{k,0}} \varsigma_j 2 m_t^{j,l} d\psi + \frac{a_e^2}{4\varsigma_j} \int_0^{m_t^{k,0}} \varsigma_j^2 4 (m_t^{j,l})^2 d\psi
\end{aligned}$$

Note that, proceeding as before by using the definition of $m_t^{j,l}$:

$$\begin{aligned}
&\int_0^{m_t^{k,0}} 2\varsigma_j m_t^{j,l} d\psi \\
=&\left(\frac{1}{2} b_e^2 f_k^2 \left(m_t^{k,1}, m_t^{k,0}\right) + 2\varsigma_j m_t^{j,h}\right) \int_0^{m_t^{k,0}} 1 d\psi \\
&+ k_1 \int_0^{m_t^{k,0}} f_k \left(m_t^{k,1}, \psi\right) d\psi \\
&- k_0 \int_0^{m_t^{k,0}} f_k \left(m_t^{k,0}, \psi\right) d\psi
\end{aligned}$$

And that

$$\begin{aligned}
&\int_0^{m_t^{k,0}} 4\varsigma_j^2 (m_t^{j,l})^2 d\psi \\
=&\left(\frac{1}{2} b_e^2 f_k^2 \left(m_t^{k,1}, m_t^{k,0}\right) + 2\varsigma_j m_t^{j,h}\right)^2 \int_0^{m_t^{k,0}} 1 d\psi \\
&+ k_1^2 \int_0^{m_t^{k,0}} f_k^2 \left(m_t^{k,1}, \psi\right) d\psi \\
&+ k_0^2 \int_0^{m_t^{k,0}} f_k^2 \left(m_t^{k,0}, \psi\right) d\psi \\
&+ 2k_1 \left(\frac{1}{2} b_e^2 f_k^2 \left(m_t^{k,1}, m_t^{k,0}\right) + 2\varsigma_j m_t^{j,h}\right) \int_0^{m_t^{k,0}} f_k \left(m_t^{k,1}, \psi\right) d\psi \\
&- 2k_0 \left(\frac{1}{2} b_e^2 f_k^2 \left(m_t^{k,1}, m_t^{k,0}\right) + 2\varsigma_j m_t^{j,h}\right) \int_0^{m_t^{k,0}} f_k \left(m_t^{k,0}, \psi\right) d\psi \\
&- 2k_1 k_0 \int_0^{m_t^{k,0}} f_k \left(m_t^{k,1}, \psi\right) f_k \left(m_t^{k,0}, \psi\right) d\psi
\end{aligned}$$

Collecting terms, we get:

$$
\begin{aligned}
&\int_0^{m_t^{k,0}} \eta m_t^{j,l} + a_e^2 \varsigma_j (m_t^{j,l})^2 d\psi \\
=&\left(\eta m_t^{j,h} + a_e^2 \varsigma_j (m_t^{j,h})^2 + \frac{a_e^2}{16\varsigma_j} b_e^4 f_k^4(m_t^{k,1}, m_t^{k,0})\right) \int_0^{m_t^{k,0}} 1 d\psi \\
&+ \left(\frac{b_e^2 f_k^2(m_t^{k,1}, m_t^{k,0})}{4\varsigma_j} \left(\eta + 2a_e^2 \varsigma_j m_t^{j,h}\right)\right) \int_0^{m_t^{k,0}} 1 d\psi \\
&+ \left(\frac{k_1}{2\varsigma_j} \left(\eta + 2a_e^2 \varsigma_j m_t^{j,h}\right) + \frac{a_e^2}{4\varsigma_j} k_1 b_e^2 f_k^2(m_t^{k,1}, m_t^{k,0})\right) \int_0^{m_t^{k,0}} f_k(m_t^{k,1}, \psi) d\psi \\
&- \left(\frac{k_0}{2\varsigma_j} \left(\eta + 2a_e^2 \varsigma_j m_t^{j,h}\right) + \frac{a_e^2}{4\varsigma_j} k_0 b_e^2 f_k^2(m_t^{k,1}, m_t^{k,0})\right) \int_0^{m_t^{k,0}} f_k(m_t^{k,0}, \psi) d\psi \\
&- \frac{2a_e^2 k_1 k_0}{4\varsigma_j} \int_0^{m_t^{k,0}} f_k\left(m_t^{k,1}, \psi\right) f_k\left(m_t^{k,0}, \psi\right) d\psi \\
&+ \frac{a_e^2 k_1^2}{4\varsigma_j} \int_0^{m_t^{k,0}} f_k^2\left(m_t^{k,1}, \psi\right) d\psi \\
&+ \frac{a_e^2 k_0^2}{4\varsigma_j} \int_0^{m_t^{k,0}} f_k^2\left(m_t^{k,0}, \psi\right) d\psi
\end{aligned}
$$

Which is exact, since all the integrals have been evaluated analytically previously.

### A.2.5 integral of $p\left(e_t^{j,l}\right) q\left(1, e_t^{k,1}\right)$

This integral is one part of the expression for $H_t^1$, being households with low credit aversion obtaining both a job and credit.

$$
\begin{aligned}
&\int_0^{m_t^{j,l}} \int_0^{m_t^{k,0}} p\left(e_t^{j,l}\right) q\left(1, e_t^{k,1}\right) d\psi d\phi \\
=&\int_0^{m_t^{k,0}} q\left(1, e_t^{k,1}\right) \left(\int_0^{m_t^{j,l}} p\left(e_t^{j,l}\right) d\phi\right) d\psi
\end{aligned}
$$

Note that $q\left(1, e_t^{k,1}\right)$ is independent of $\phi$, and we have already calculated the integral of $p\left(e_t^{j,l}\right)$, so we obtain an approximation in the general case given by:

$$
\begin{aligned}
=&\int_0^{m_t^{k,0}} q\left(1, e_t^{k,1}\right)\left(\eta m_t^{j,l} + a_e^2 \varsigma_j \sigma_j \left(m_t^{j,l}\right)^{1+\sigma_j}\right) d\psi \\
\approx&\frac{\eta m_t^{k,0}}{2}\left(q\left(1, e_t^{k,1}|_0\right) m_t^{j,l}|_0 + q\left(1, e_t^{k,1}|_{m_t^{k,0}}\right) m_t^{j,l}|_{m_t^{k,0}}\right) \\
&+ \frac{a_e^2 \varsigma_j \sigma_j m_t^{k,0}}{2}\left(q\left(1, e_t^{k,1}|_0\right)\left(m_t^{j,l}|_0\right)^{1+\sigma_j} + q\left(1, e_t^{k,1}|_{m_t^{k,0}}\right)\left(m_t^{j,l}|_{m_t^{k,0}}\right)^{1+\sigma_j}\right)
\end{aligned}
$$

Or, if we set $\sigma_j = 1$, then:

$$
\begin{aligned}
&\int_0^{m_t^{k,0}} q\left(1, e_t^{k,1}\right)\left(\eta m_t^{j,l} + a_e^2 \varsigma_j \left(m_t^{j,l}\right)^2\right) d\psi \\
=&\int_0^{m_t^{k,0}} \left(k_1 + b_e^2 f_k(m_t^{k,1}, \psi)\right)\left(\eta m_t^{j,l} + a_e^2 \varsigma_j \left(m_t^{j,l}\right)^2\right) d\psi \\
=&k_1 \int_0^{m_t^{k,0}} \left(\eta m_t^{j,l} + a_e^2 \varsigma_j \left(m_t^{j,l}\right)^2\right) d\psi + \frac{b_e^2 \eta}{2\varsigma_j}\int_0^{m_t^{k,0}} f_k(m_t^{k,1}, \psi) 2\varsigma_j m_t^{j,l} d\psi \\
&+ \frac{b_e^2 a_e^2}{4\varsigma_j}\int_0^{m_t^{k,0}} f_k(m_t^{k,1}, \psi) 4\varsigma_j^2 (m_t^{j,l})^2 d\psi
\end{aligned}
$$

The integral in the first term has been dealt with above. So thus we need to examine the last two integrals:

$$
\begin{aligned}
&\int_0^{m_t^{k,0}} f_k(m_t^{k,1}, \psi) 2\varsigma_j m_t^{j,l} d\psi \\
=&\left(\frac{1}{2} b_e^2 f_k^2\left(m_t^{k,1}, m_t^{k,0}\right) + 2\varsigma_j m_t^{j,h}\right)\int_0^{m_t^{k,0}} f_k(m_t^{k,1}, \psi) d\psi \\
&+ k_1 \int_0^{m_t^{k,0}} f_k^2\left(m_t^{k,1}, \psi\right) d\psi \\
&- k_0 \int_0^{m_t^{k,0}} f_k(m_t^{k,1}, \psi) f_k\left(m_t^{k,0}, \psi\right) d\psi
\end{aligned}
$$

along with

$$
\begin{aligned}
&\int_0^{m_t^{k,0}} f_k(m_t^{k,1},\psi)4\varsigma_j^2(m_t^{j,l})^2 d\psi \\
=&\left(\frac{1}{2}b_e^2 f_k^2\left(m_t^{k,1},m_t^{k,0}\right)+2\varsigma_j m_t^{j,h}\right)^2\int_0^{m_t^{k,0}} f_k(m_t^{k,1},\psi)d\psi \\
&+k_1^2\int_0^{m_t^{k,0}} f_k^3\left(m_t^{k,1},\psi\right)d\psi \\
&+k_0^2\int_0^{m_t^{k,0}} f_k(m_t^{k,1},\psi)f_k^2\left(m_t^{k,0},\psi\right)d\psi \\
&+2k_1\left(\frac{1}{2}b_e^2 f_k^2\left(m_t^{k,1},m_t^{k,0}\right)+2\varsigma_j m_t^{j,h}\right)\int_0^{m_t^{k,0}} f_k^2\left(m_t^{k,1},\psi\right)d\psi \\
&-2k_0\left(\frac{1}{2}b_e^2 f_k^2\left(m_t^{k,1},m_t^{k,0}\right)+2\varsigma_j m_t^{j,h}\right)\int_0^{m_t^{k,0}} f_k(m_t^{k,1},\psi)f_k\left(m_t^{k,0},\psi\right)d\psi \\
&-2k_1k_0\int_0^{m_t^{k,0}} f_k^2\left(m_t^{k,1},\psi\right)f_k\left(m_t^{k,0},\psi\right)d\psi
\end{aligned}
$$

Again, we can collect all the terms from the three integrals, but the expressions are beginning to get sufficiently large that no particular insight is gained from writing them out.

### A.2.6 integral of $p\left(e_t^{j,l}\right)q\left(0,e_t^{k,0}\right)$

The integral below relates to the size of $Un_t^1$. We proceed as previously:

$$\int_0^{m_t^{j,l}}\int_0^{m_t^{k,0}} p\left(e_t^{j,l}\right)q\left(0,e_t^{k,0}\right)d\psi d\phi$$

Note that $q\left(0,e_t^{k,0}\right)$ is independent of $\phi$, so as before we obtain:

$$
\begin{aligned}
=&\int_0^{m_t^{k,0}} q\left(0,e_t^{k,0}\right)m_t^{j,l}\left(\eta+a_e^2\varsigma_j\sigma_j\left(m_t^{j,l}\right)^{\sigma_j}\right)d\psi \\
\approx&\frac{\eta m_t^{k,0}}{2}\left(q\left(0,e_t^{k,0}|_0\right)m_t^{j,l}|_0+k_0 m_t^{j,l}|_{m_t^{k,0}}\right) \\
&+\frac{a_e^2\varsigma_j\sigma_j m_t^{k,0}}{2}\left(q\left(0,e_t^{k,0}|_0\right)\left(m_t^{j,l}|_0\right)^{1+\sigma_j}+k_0\left(m_t^{j,l}|_{m_t^{k,0}}\right)^{1+\sigma_j}\right)
\end{aligned}
$$

While if we set $\sigma_j=1$, we get:

$$\begin{aligned}
&\int_0^{m_t^{k,0}} q\left(0, e_t^{k,0}\right)\left(\eta m_t^{j,l} + a_e^2 \varsigma_j \left(m_t^{j,l}\right)^2\right) d\psi \\
=&\int_0^{m_t^{k,0}} \left(k_0 + b_e^2 f_k(m_t^{k,0}, \psi)\right)\left(\eta m_t^{j,l} + a_e^2 \varsigma_j \left(m_t^{j,l}\right)^2\right) d\psi \\
=&k_0 \int_0^{m_t^{k,0}} \left(\eta m_t^{j,l} + a_e^2 \varsigma_j \left(m_t^{j,l}\right)^2\right) d\psi + \frac{b_e^2 \eta}{2\varsigma_j} \int_0^{m_t^{k,0}} f_k(m_t^{k,0}, \psi) 2\varsigma_j m_t^{j,l} d\psi \\
&+ \frac{b_e^2 a_e^2}{4\varsigma_j} \int_0^{m_t^{k,0}} f_k(m_t^{k,0}, \psi) 4\varsigma_j^2 (m_t^{j,l})^2 d\psi
\end{aligned}$$

As before, the first integral has already been examined. The last two integrals will be examined below:

$$\begin{aligned}
&\int_0^{m_t^{k,0}} f_k(m_t^{k,0}, \psi) 2\varsigma_j m_t^{j,l} d\psi \\
=&\left(\frac{1}{2} b_e^2 f_k^2\left(m_t^{k,1}, m_t^{k,0}\right) + 2\varsigma_j m_t^{j,h}\right) \int_0^{m_t^{k,0}} f_k(m_t^{k,0}, \psi) d\psi \\
&+ k_1 \int_0^{m_t^{k,0}} f_k(m_t^{k,1}, \psi) f_k\left(m_t^{k,0}, \psi\right) d\psi \\
&- k_0 \int_0^{m_t^{k,0}} f_k^2\left(m_t^{k,0}, \psi\right) d\psi
\end{aligned}$$

along with

$$
\begin{aligned}
&\int_0^{m_t^{k,0}} f_k(m_t^{k,0},\psi)4\varsigma_j^2(m_t^{j,l})^2 d\psi \\
=&\left(\frac{1}{2}b_e^2 f_k^2\left(m_t^{k,1},m_t^{k,0}\right)+2\varsigma_j m_t^{j,h}\right)^2\int_0^{m_t^{k,0}} f_k(m_t^{k,0},\psi)d\psi \\
&+k_1^2\int_0^{m_t^{k,0}} f_k(m_t^{k,0},\psi)f_k^2\left(m_t^{k,1},\psi\right)d\psi \\
&+k_0^2\int_0^{m_t^{k,0}} f_k^3\left(m_t^{k,0},\psi\right)d\psi \\
&+2k_1\left(\frac{1}{2}b_e^2 f_k^2\left(m_t^{k,1},m_t^{k,0}\right)+2\varsigma_j m_t^{j,h}\right)\int_0^{m_t^{k,0}} f_k(m_t^{k,0},\psi)f_k\left(m_t^{k,1},\psi\right)d\psi \\
&-2k_0\left(\frac{1}{2}b_e^2 f_k^2\left(m_t^{k,1},m_t^{k,0}\right)+2\varsigma_j m_t^{j,h}\right)\int_0^{m_t^{k,0}} f_k^2\left(m_t^{k,0},\psi\right)d\psi \\
&-2k_1k_0\int_0^{m_t^{k,0}} f_k^2\left(m_t^{k,0},\psi\right)f_k\left(m_t^{k,1},\psi\right)d\psi
\end{aligned}
$$

Again, we will need to collect the common terms, but as above, no particular insight is obtained from this.

### A.2.7 integral of $p\left(e_t^{j,m}\right)q\left(1,e_t^{k,1}\right)$

This is the other part of $H_t^1$, households with medium credit aversion that manage to obtain a job and credit.

$$
\int_0^{m_t^{j,m}}\int_{m_t^{k,0}}^{m_t^{k,1}} p\left(e_t^{j,m}\right)q\left(1,e_t^{k,1}\right)d\psi d\phi
$$

Following the same reasoning as above, we obtain:

$$
\begin{aligned}
=&\int_{m_t^{k,0}}^{m_t^{k,1}} q\left(1,e_t^{k,1}\right)m_t^{j,m}\left(\eta+a_e^2\varsigma_j\sigma_j\left(m_t^{j,m}\right)^{\sigma_j}\right)d\psi \\
\approx&(m_t^{k,1}-m_t^{k,0})\frac{\eta}{2}\left(k_1 m_t^{j,m}|_{m_t^{k,1}}+q\left(1,e_t^{k,1}|_{m_t^{k,0}}\right)m_t^{j,m}|_{m_t^{k,0}}\right) \\
&+(m_t^{k,1}-m_t^{k,0})\frac{a_e^2\varsigma_j\sigma_j}{2}\left(k_1\left(m_t^{j,m}|_{m_t^{k,1}}\right)^{1+\sigma_j}+q\left(1,e_t^{k,1}|_{m_t^{k,0}}\right)\left(m_t^{j,m}|_{m_t^{k,0}}\right)^{1+\sigma_j}\right)
\end{aligned}
$$

Or if $\sigma_j = 1$, then we get:

$$\begin{aligned}
&\int_{m_t^{k,0}}^{m_t^{k,1}} q\left(1, e_t^{k,1}\right)\left(\eta m_t^{j,m} + a_e^2 \varsigma_j \left(m_t^{j,m}\right)^2\right) d\psi \\
=&k_1 \int_{m_t^{k,0}}^{m_t^{k,1}} \left(\eta m_t^{j,m} + a_e^2 \varsigma_j \left(m_t^{j,m}\right)^2\right) d\psi + \frac{b_e^2 \eta}{2\varsigma_j} \int_{m_t^{k,0}}^{m_t^{k,1}} f_k(m_t^{k,1}, \psi) 2\varsigma_j m_t^{j,m} d\psi \\
&+ \frac{b_e^2 a_e^2}{4\varsigma_j} \int_{m_t^{k,0}}^{m_t^{k,1}} f_k(m_t^{k,1}, \psi) 4\varsigma_j^2 (m_t^{j,m})^2 d\psi
\end{aligned}$$

As before, examining the last two integrals, we get:

$$\begin{aligned}
&\int_{m_t^{k,0}}^{m_t^{k,1}} f_k(m_t^{k,1}, \psi) \varsigma_j 2 m_t^{j,m} d\psi \\
=&\int_{m_t^{k,0}}^{m_t^{k,1}} k_1 f_k^2\left(m_t^{k,1}, \psi\right) + \frac{1}{2} b_e^2 f_k^3\left(m_t^{k,1}, \psi\right) + \varsigma_j 2 m_t^{j,h} f_k(m_t^{k,1}, \psi) d\psi \\
=&\varsigma_j 2 m_t^{j,h} \int_{m_t^{k,0}}^{m_t^{k,1}} f_k(m_t^{k,1}, \psi) d\psi \\
&+ k_1 \int_{m_t^{k,0}}^{m_t^{k,1}} f_k^2\left(m_t^{k,1}, \psi\right) d\psi \\
&+ \frac{1}{2} b_e^2 \int_{m_t^{k,0}}^{m_t^{k,1}} f_k^3\left(m_t^{k,1}, \psi\right) d\psi
\end{aligned}$$

And then the more intricate integral given by:

$$
\begin{aligned}
&\int_{m_t^{k,0}}^{m_t^{k,1}} f_k(m_t^{k,1},\psi)\varsigma_j^2 4(m_t^{j,m})^2 d\psi \\
=&\int_{m_t^{k,0}}^{m_t^{k,1}} f_k(m_t^{k,1},\psi)\left(k_1 f_k\left(m_t^{k,1},\psi\right)+\frac{1}{2}b_e^2 f_k^2\left(m_t^{k,1},\psi\right)+\varsigma_j 2m_t^{j,h}\right)^2 d\psi \\
=&\int_{m_t^{k,0}}^{m_t^{k,1}} \left(k_1^2+2b_e^2\varsigma_j m_t^{j,h}\right) f_k^3\left(m_t^{k,1},\psi\right)+\frac{1}{4}b_e^4 f_k^5\left(m_t^{k,1},\psi\right) \\
&\quad+\varsigma_j^2 4(m_t^{j,h})^2 f_k(m_t^{k,1},\psi)+k_1 b_e^2 f_k^4\left(m_t^{k,1},\psi\right)+4k_1\varsigma_j m_t^{j,h} f_k^2\left(m_t^{k,1},\psi\right) d\psi \\
=&\varsigma_j^2 4(m_t^{j,h})^2\int_{m_t^{k,0}}^{m_t^{k,1}} f_k(m_t^{k,1},\psi)d\psi \\
&+4k_1\varsigma_j m_t^{j,h}\int_{m_t^{k,0}}^{m_t^{k,1}} f_k^2\left(m_t^{k,1},\psi\right)d\psi \\
&+\left(k_1^2+2b_e^2\varsigma_j m_t^{j,h}\right)\int_{m_t^{k,0}}^{m_t^{k,1}} f_k^3\left(m_t^{k,1},\psi\right)d\psi \\
&+k_1 b_e^2\int_{m_t^{k,0}}^{m_t^{k,1}} f_k^4\left(m_t^{k,1},\psi\right)d\psi \\
&+\frac{1}{4}b_e^4\int_{m_t^{k,0}}^{m_t^{k,1}} f_k^5\left(m_t^{k,1},\psi\right)d\psi
\end{aligned}
$$

Again, we simply note that we will need to sum the three integrals together, but no particular insight is gained by writing it out.

## A.3 The implementation in GiNaC

The actual calculation of the integrals has been done in a computer algebra system. For the reduced model, we have the following source code, written in C++:

```
//Here we define a few useful shorthand functions and results.

ex fk(ex var1,ex var2)
{
  return vsigmak * 2 *( var1 - var2 );
```

```
}

ex fj(ex var1,ex var2)
{
   return vsigmaj * 2 *( var1 - var2 );
}

ex q1(ex point)
{
   return k1 + pow ( be , 2 ) * fk(mk1,point);
}


//This is the marginal household with medium credit aversion,
//evaluated in a point between mk1 and mk0.

ex mjm(ex point)
{
   ex i =
   (
      k1 * fk(mk1,point)
      + ( pow(be,2) / 2 ) * pow( fk(mk1,point) , 2)
      + vsigmaj * 2 * mjh
   ) / ( vsigmaj * 2 );

   return i ;
}

//These are the probabilities for the three types of households
//of obtaining a job:

ex ph(ex phi,ex psi)
{
   return n + pow ( ae , 2 ) * fj(mjh,phi);
}

ex pm(ex phi,ex psi)
{
   return n + pow ( ae , 2 ) * fj(mjm(psi),phi);
}
```

```
//The building blocks for evaluating the integrals.
//The final number at the end of the function name
//denotes the power to which fj/fk is raised.

ex intfjh1(ex low, ex high)
{
   possymbol point ("point");
   ex a = mjh * point - pow(point,2)/2;

   ex i = vsigmaj * 2 * (
      a.subs(point==high) - a.subs(point==low)
      );
   return i ;
}

ex intfjh2(ex low, ex high)
{
   possymbol point ("point");
   ex a = pow(mjh,2) * point
       - mjh * pow(point,2) + pow(point,3)/3;

   ex i = pow(vsigmaj,2) * 4 * (
       a.subs(point==high) - a.subs(point==low)
       );
   return i ;
}

ex int1(ex low, ex high)
{
   possymbol point ("point");
   ex a = point;

   ex i =   a.subs(point==high) - a.subs(point==low);
   return i ;
}


ex intfk1(ex var, ex low, ex high)
{
   possymbol point ("point");
```

```
    ex a = var * point - pow(point,2)/(2);

    ex i = vsigmak * 2 * (
        a.subs(point==high) - a.subs(point==low)
        );
    return i ;
}

ex intfk2(ex var, ex low, ex high)
{
    possymbol point ("point");
    ex a =
          pow(var,2) * point
    - 2* pow(var,1) * pow(point,2)/(2)
    +                  pow(point,3)/(3);

    ex i = pow(vsigmak * 2,2) * (
        a.subs(point==high) - a.subs(point==low)
        );
    return i ;
}


ex intfk3(ex var, ex low, ex high)
{
    possymbol point ("point");
    ex a =
          pow(var,3) * point
    - 3* pow(var,2) * pow(point,2)/(2)
    + 3* pow(var,1) * pow(point,3)/(3)
    -                 pow(point,4)/(4);

    ex i = pow(vsigmak * 2,3) * (
        a.subs(point==high) - a.subs(point==low)
        );
    return i ;
}

ex intfk4(ex var, ex low, ex high)
{
    possymbol point ("point");
```

```
    ex a =
          pow(var,4) * point
    - 4* pow(var,3) * pow(point,2)/(2)
    + 6* pow(var,2) * pow(point,3)/(3)
    - 4* pow(var,1) * pow(point,4)/(4)
    +                 pow(point,5)/(5);

    ex i = pow(vsigmak * (2),4) * (
       a.subs(point==high) - a.subs(point==low)
       );
    return i ;
}

ex intfk5(ex var, ex low, ex high)
{
    possymbol point ("point");
    ex a =
          pow(var,5) * point
    - 5 *pow(var,4) * pow(point,2)/(2)
    + 10*pow(var,3) * pow(point,3)/(3)
    - 10*pow(var,2) * pow(point,4)/(4)
    + 5 *pow(var,1) * pow(point,5)/(5)
    -                 pow(point,6)/(6);

    ex i = pow(vsigmak * 2,5) * (
       a.subs(point==high) - a.subs(point==low)
       );
    return i ;
}

ex intfk8(ex var, ex low, ex high)
{
    possymbol point ("point");
    ex a =
          pow(var,8) * point
    - 8 *pow(var,7) * pow(point,2)/(2)
    + 28*pow(var,6) * pow(point,3)/(3)
    - 56*pow(var,5) * pow(point,4)/(4)
    + 70*pow(var,4) * pow(point,5)/(5)
    - 56*pow(var,3) * pow(point,6)/(6)
    + 28*pow(var,2) * pow(point,7)/(7)
```

```
  - 8 *pow(var,1) * pow(point,8)/(8)
  +                 pow(point,9)/(9);

  ex i = pow(vsigmak * 2,8) * (
     a.subs(point==high) - a.subs(point==low)
     );
  return i ;
}

ex intfkx1(ex var1, ex var2, ex low, ex high)
{
  possymbol point ("point");
  ex a =
        pow(var1,1) *    pow(var2,1) * point
  -     pow(var1,1)                  * pow(point,2)/(2)
  -                      pow(var2,1) * pow(point,2)/(2)
  +                                    pow(point,3)/(3);

  ex i = pow(vsigmak * 2,2) * (
     a.subs(point==high) - a.subs(point==low)
     );
  return i ;
}


ex intfkx2(ex var1, ex var2, ex low, ex high)
{
  possymbol point ("point");
  ex a =
        pow(var1,2) * pow(var2,1) * point
  - 2* pow(var1,2)                * pow(point,2)/(2)
  - 2* pow(var1,1) * pow(var2,1)  * pow(point,2)/(2)
  + 2* pow(var1,1)                * pow(point,3)/(3)
  +                  pow(var2,1)  * pow(point,3)/(3)
  -                                 pow(point,4)/(4);

  ex i = pow(vsigmak * 2,3) * (
     a.subs(point==high) - a.subs(point==low)
     );
  return i ;
}
```

```
ex intmjm()
{
   ex a = k1;
   ex b = pow(be,2)/2;
   ex c = fj(mjh,0);

   ex i =
      a * intfk1(mk1,0,mk1)
   + b * intfk2(mk1,0,mk1)
   + c * int1(0,mk1);

   return i ;
}


ex intmjm2()
{
   ex a = k1;
   ex b = pow(be,2)/2;
   ex c = fj(mjh,0);

   ex i =
          pow(a,2) * intfk2(mk1,0,mk1)
   +      pow(b,2) * intfk4(mk1,0,mk1)
   +      pow(c,2) * int1(0,mk1)
   + 2 * a * b     * intfk3(mk1,0,mk1)
   + 2 * a * c     * intfk1(mk1,0,mk1)
   + 2 * b * c     * intfk2(mk1,0,mk1);

   return i ;
}


ex intmjm3()
{
   ex a = k1;
   ex b = pow(be,2)/2;
   ex c = fj(mjh,0);
```

```
    ex i =
              pow(a,3)        * intfk3(mk1,0,mk1)
    +         pow(b,3)        * intfk8(mk1,0,mk1)
    +         pow(c,3)        * int1(0,mk1)
    + 3 * pow(a,2) * b * intfk4(mk1,0,mk1)
    + 3 * pow(a,2) * c * intfk2(mk1,0,mk1)
    + 3 * a * pow(b,2) * intfk5(mk1,0,mk1)
    + 3 * a * pow(c,2) * intfk1(mk1,0,mk1)
    + 6 * a * b * c      * intfk3(mk1,0,mk1)
    + 3 * pow(b,2) * c * intfk4(mk1,0,mk1)
    + 3 * b * pow(c,2) * intfk2(mk1,0,mk1);

    return i ;
}


ex intfk1mjm()
{
    ex a = k1;
    ex b = pow(be,2)/2;
    ex c = fj(mjh,0);

    ex i =
       c * intfk1(mk1,0,mk1)
    + a * intfk2(mk1,0,mk1)
    + b * intfk3(mk1,0,mk1);

    return i ;
}


ex intfk1mjm2()
{
    ex a = k1;
    ex b = pow(be,2)/2;
    ex c = fj(mjh,0);

    ex i =
              pow(c,2) * intfk1(mk1,0,mk1)
    +         pow(a,2) * intfk3(mk1,0,mk1)
    +         pow(b,2) * intfk5(mk1,0,mk1)
```

```
    + 2 * a * b     * intfk4(mk1,0,mk1)
    + 2 * a * c     * intfk2(mk1,0,mk1)
    + 2 * b * c     * intfk3(mk1,0,mk1);

    return i ;
}

//So having now written out all the integrals,
// we can obtain the employment levels:

lst employment()
{

    ex intph = ( 1 - mk1 )
        * ( n * mjh + vsigmaj * pow(ae,2) * pow(mjh,2) );

    ex intpm =
    (
          n/(2 * vsigmaj) * intmjm()
        + pow(ae,2)/(4 * vsigmaj) * intmjm2()
    );

    ex intq1pm =
    (
          k1 * intpm
        + (pow(be,2)*n)/(2 * vsigmaj) * intfk1mjm()
        + (pow(be,2)*pow(ae,2))/(4 * vsigmaj)
              * intfk1mjm2()
    );

    ex H1v = intq1pm;
    ex H0v = intpm + intph - H1v;
    return lst(H1v, H0v);
}
```

# Appendix B

# Integrating the utility function

## B.1 The components of the utility function

The utility function for the family in the full model is given by a sum of integrals:

$$
\begin{aligned}
U =& \int_0^{m_t^{j,l}} \int_0^{m_t^{k,0}} U_L d\psi d\phi + \int_0^{m_t^{j,m}} \int_{m_t^{k,0}}^{m_t^{k,1}} U_M d\psi d\phi + \int_0^{m_t^{j,h}} \int_{m_t^{k,1}}^{1} U_H d\psi d\phi \\
&+ \int_{m_t^{j,l}}^{1} \int_0^{m_t^{k,0}} U_{An} d\psi d\phi + \int_{m_t^{j,m}}^{1} \int_{m_t^{k,0}}^{m_t^{k,1}} \log c_t(0,0) d\psi d\phi + \int_{m_t^{j,h}}^{1} \int_{m_t^{k,1}}^{1} \log c_t(0,0) d\psi d\phi \\
=& \int_0^{m_t^{j,l}} \int_0^{m_t^{k,0}} \eta f_j(m_t^{j,l}, \phi) + \frac{1}{2} a_e^2 f_j^2(m_t^{j,l}, \phi) d\psi d\phi \\
&+ \int_0^{m_t^{j,m}} \int_{m_t^{k,0}}^{m_t^{k,1}} \eta f_j(m_t^{j,m}, \phi) + \frac{1}{2} a_e^2 f_j^2(m_t^{j,m}, \phi) d\psi d\phi \\
&+ \int_0^{m_t^{j,h}} \int_{m_t^{k,1}}^{1} \eta f_j(m_t^{j,h}, \phi) + \frac{1}{2} a_e^2 f_j^2(m_t^{j,h}, \phi) d\psi d\phi \\
&+ \int_0^{1} \int_0^{m_t^{k,0}} k_0 f_k(m_t^{k,0}, \psi) + \frac{1}{2} b_e^2 f_k^2(m_t^{k,0}, \psi) d\psi d\phi \\
&+ \int_0^{1} \int_0^{1} \log c_t(0,0) d\psi d\phi
\end{aligned}
$$

Where the equality comes from noticing a few common parts in each of the ex-ante utility components.

This means we have five integrals to evaluate, of which the last one is trivial: $\int_0^1 \int_0^1 \log c_t(0,0) d\psi d\phi = \log c_t(0,0)$

### B.1.1 Integrating the high credit aversion households

The integral under consideration here is almost identical to the integral in [Christiano et al., 2010]. In this case it is simply a matter of expanding the product, and term by term integration.

We obtain:

$$\begin{aligned}
&\int_0^{m_t^{j,h}} \int_{m_t^{k,1}}^1 \eta f_j(m_t^{j,h},\phi) + \frac{1}{2}a_e^2 f_j^2(m_t^{j,h},\phi) d\psi d\phi \\
=&\int_{m_t^{k,1}}^1 \eta \varsigma_j \sigma_j (m_t^{j,h})^{1+\sigma_j} + \frac{1}{2}a_e^2\varsigma_j^2 \left(\frac{\sigma_j^2(1+\sigma_j)(m_t^{j,h})^{1+2\sigma_j}}{1+2\sigma_j}\right) d\psi \\
=&(1-m_t^{k,1})\left(\eta \varsigma_j \sigma_j (m_t^{j,h})^{1+\sigma_j} + \frac{1}{2}a_e^2\varsigma_j^2 \left(\frac{\sigma_j^2(1+\sigma_j)(m_t^{j,h})^{1+2\sigma_j}}{1+2\sigma_j}\right)\right)
\end{aligned}$$

### B.1.2 Integrating the credit market integral

The penultimate term in the utility function is the only one directly to contain $\psi$, but we note that the structure is analogous to the integral considered above, so we get:

$$\begin{aligned}
&\int_0^1 \int_0^{m_t^{k,0}} k_0 f_k(m_t^{k,0},\psi) + \frac{1}{2}b_e^2 f_k^2(m_t^{k,0},\psi) d\psi d\phi \\
=&\int_0^1 k_0 \varsigma_k \sigma_k (m_t^{k,0})^{1+\sigma_k} + \frac{1}{2}b_e^2\varsigma_k^2 \left(\frac{\sigma_k^2(1+\sigma_k)(m_t^{k,0})^{1+2\sigma_k}}{1+2\sigma_k}\right) d\phi \\
=&k_0 \varsigma_k \sigma_k (m_t^{k,0})^{1+\sigma_k} + \frac{1}{2}b_e^2\varsigma_k^2 \left(\frac{\sigma_k^2(1+\sigma_k)(m_t^{k,0})^{1+2\sigma_k}}{1+2\sigma_k}\right)
\end{aligned}$$

### B.1.3 Integrating the medium credit aversion households

For households with medium credit aversion, we have to deal with the following integral: $\int_0^{m_t^{j,m}} \int_{m_t^{k,0}}^{m_t^{k,1}} \eta f_j(m_t^{j,m},\phi) + \frac{1}{2}a_e^2 f_j^2(m_t^{j,m},\phi) d\psi d\phi$. Note that $m_t^{j,m}$ is a function of $\psi$.

Thus, we integrate with respect to $\phi$ first:

$$
\begin{aligned}
&\int_0^{m_t^{j,m}} \int_{m_t^{k,0}}^{m_t^{k,1}} \eta f_j(m_t^{j,m},\phi) + \frac{1}{2} a_e^2 f_j^2(m_t^{j,m},\phi) d\psi d\phi \\
=&\int_{m_t^{k,0}}^{m_t^{k,1}} \eta \varsigma_j \sigma_j (m_t^{j,m})^{1+\sigma_j} + \frac{1}{2} a_e^2 \varsigma_j^2 \left( \frac{\sigma_j^2 (1+\sigma_j)(m_t^{j,m})^{1+2\sigma_j}}{1+2\sigma_j} \right) d\psi \\
\approx&(m_t^{k,1} - m_t^{k,0}) \left( \frac{\eta \varsigma_j \sigma_j}{2} \left( (m_t^{j,m})^{1+\sigma_j}_{|m_t^{k,1}} + (m_t^{j,m})^{1+\sigma_j}_{|m_t^{k,0}} \right) \right) \\
&+ (m_t^{k,1} - m_t^{k,0}) \left( \frac{1}{4} a_e^2 \varsigma_j^2 \frac{\sigma_j^2 (1+\sigma_j)}{1+2\sigma_j} \left( (m_t^{j,m})^{1+2\sigma_j}_{|m_t^{k,1}} + (m_t^{j,m})^{1+2\sigma_j}_{|m_t^{k,0}} \right) \right)
\end{aligned}
$$

Where the last step is a rough numerical integration.

But if $\sigma_j = 1$, we can write out an explicit expression:

$$
\begin{aligned}
&\int_0^{m_t^{j,m}} \int_{m_t^{k,0}}^{m_t^{k,1}} \eta f_j(m_t^{j,m},\phi) + \frac{1}{2} a_e^2 f_j^2(m_t^{j,m},\phi) d\psi d\phi \\
=&\int_{m_t^{k,0}}^{m_t^{k,1}} \eta \varsigma_j (m_t^{j,m})^2 + \frac{a_e^2 \varsigma_j^2}{3} (m_t^{j,m})^3 d\psi \\
=&\frac{\eta}{4\varsigma_j} \int_{m_t^{k,0}}^{m_t^{k,1}} 4\varsigma_j^2 (m_t^{j,m})^2 d\psi + \frac{a_e^2}{24\varsigma_j} \int_{m_t^{k,0}}^{m_t^{k,1}} 8\varsigma_j^3 (m_t^{j,m})^3 d\psi
\end{aligned}
$$

The first integral has been dealt with in the previous section. The second integral is expanded below:

$$
\begin{aligned}
&\int_{m_t^{k,0}}^{m_t^{k,1}} 8\varsigma_j^3(m_t^{j,m})^3 d\psi \\
=&k_1^3 \int_{m_t^{k,0}}^{m_t^{k,1}} f_k^3(m_t^{k,1},\psi)d\psi \\
&+ 3k_1^2\frac{1}{2}b_e^2 \int_{m_t^{k,0}}^{m_t^{k,1}} f_k^4(m_t^{k,1},\psi)d\psi \\
&+ 3k_1^2 2\varsigma_j m_t^{j,h} \int_{m_t^{k,0}}^{m_t^{k,1}} f_k^2(m_t^{k,1},\psi)d\psi \\
&+ 3k_1\frac{1}{4}b_e^4 \int_{m_t^{k,0}}^{m_t^{k,1}} f_k^5(m_t^{k,1},\psi)d\psi \\
&+ 6k_1\frac{1}{2}b_e^2 4\varsigma_j(m_t^{j,h})^2 \int_{m_t^{k,0}}^{m_t^{k,1}} f_k^3(m_t^{k,1},\psi)d\psi \\
&+ 3k_1 4\varsigma_j^2(m_t^{j,h})^2 \int_{m_t^{k,0}}^{m_t^{k,1}} f_k(m_t^{k,1},\psi)d\psi \\
&+ \frac{1}{8}b_e^8 \int_{m_t^{k,0}}^{m_t^{k,1}} f_k^8(m_t^{k,1},\psi)d\psi \\
&+ 3\frac{1}{4}b_e^4 2\varsigma_j m_t^{j,h} \int_{m_t^{k,0}}^{m_t^{k,1}} f_k^4(m_t^{k,1},\psi)d\psi \\
&+ 3\frac{1}{2}b_e^2 4\varsigma_j^2(m_t^{j,h})^2 \int_{m_t^{k,0}}^{m_t^{k,1}} f_k^2(m_t^{k,1},\psi)d\psi \\
&+ 8\varsigma_j^3(m_t^{j,h})^3 \int_{m_t^{k,0}}^{m_t^{k,1}} 1d\psi
\end{aligned}
$$

### B.1.4 Integrating the low credit aversion households

For households with medium credit aversion, we have to deal with the following integral: $\int_0^{m_t^{j,l}}\int_0^{m_t^{k,0}} \eta f_j(m_t^{j,l},\phi) + \frac{1}{2}a_e^2 f_j^2(m_t^{j,l},\phi)d\psi d\phi$. Note that $m_t^{j,l}$ is a function of $\psi$.

Thus, we integrate with respect to $\phi$ first:

$$\begin{aligned}
&\int_0^{m_t^{j,l}}\int_0^{m_t^{k,0}} \eta f_j(m_t^{j,l},\phi) + \frac{1}{2}a_e^2 f_j^2(m_t^{j,l},\phi)d\psi d\phi \\
=&\int_0^{m_t^{k,0}} \eta\varsigma_j\sigma_j(m_t^{j,l})^{1+\sigma_j} + \frac{1}{2}a_e^2\varsigma_j^2\left(\frac{\sigma_j^2(1+\sigma_j)(m_t^{j,l})^{1+2\sigma_j}}{1+2\sigma_j}\right)d\psi \\
\approx&(m_t^{k,0})\frac{\eta\varsigma_j\sigma_j}{2}\left((m_t^{j,l})^{1+\sigma_j}_{|0} + (m_t^{j,l})^{1+\sigma_j}_{|m_t^{k,0}}\right) \\
&+ (m_t^{k,0})\frac{a_e^2\varsigma_j^2}{4}\frac{\sigma_j^2(1+\sigma_j)}{1+2\sigma_j}\left((m_t^{j,l})^{1+2\sigma_j}_{|0} + (m_t^{j,l})^{1+2\sigma_j}_{|m_t^{k,0}}\right)
\end{aligned}$$

Where the last step is a rough numerical integration.

But if $\sigma_j = 1$, we can write out an explicit expression:

$$\begin{aligned}
&\int_0^{m_t^{j,l}}\int_0^{m_t^{k,0}} \eta f_j(m_t^{j,l},\phi) + \frac{1}{2}a_e^2 f_j^2(m_t^{j,l},\phi)d\psi d\phi \\
=&\int_0^{m_t^{k,0}} \eta\varsigma_j(m_t^{j,l})^2 + \frac{a_e^2\varsigma_j^2}{3}(m_t^{j,l})^3 d\psi \\
=&\frac{\eta}{4\varsigma_j}\int_0^{m_t^{k,0}} 4\varsigma_j^2(m_t^{j,l})^2 d\psi + \frac{a_e^2}{24\varsigma_j}\int_0^{m_t^{k,0}} 8\varsigma_j^3(m_t^{j,l})^3 d\psi
\end{aligned}$$

Similarly to above, we expand the second integral:

$$\begin{aligned}
&\int_0^{m_t^{k,0}} 8\varsigma_j^3(m_t^{j,l})^3 d\psi \\
=&z^3\int_0^{m_t^{k,0}} 1 d\psi + 3z^2k_1\int_0^{m_t^{k,0}} f_k(m_t^{k,1},\psi)d\psi \\
&- 3z^2k_0\int_0^{m_t^{k,0}} f_k(m_t^{k,0},\psi)d\psi + 3zk_1^2\int_0^{m_t^{k,0}} f_k^2(m_t^{k,1},\psi)d\psi \\
&- 6zk_1k_0\int_0^{m_t^{k,0}} f_k(m_t^{k,1},\psi)f_k(m_t^{k,0},\psi)d\psi + 3zk_0^2\int_0^{m_t^{k,0}} f_k^2(m_t^{k,0},\psi)d\psi \\
&+ k_1^3\int_0^{m_t^{k,0}} f_k^3(m_t^{k,1},\psi)d\psi - 3k_1^2k_0\int_0^{m_t^{k,0}} f_k^2(m_t^{k,1},\psi)f_k(m_t^{k,0},\psi)d\psi \\
&+ 3k_1k_0^2\int_0^{m_t^{k,0}} f_k(m_t^{k,1},\psi)f_k^2(m_t^{k,0},\psi)d\psi - k_0^3\int_0^{m_t^{k,0}} f_k^3(m_t^{k,0},\psi)d\psi
\end{aligned}$$

where $z = \frac{1}{2}b_e^2 f_k^2(m_t^{k,1}, m_t^{k,0}) + 2\varsigma_j m_t^{j,h}$

## B.2 Bringing it all together

For general $\sigma_j$ we get:

$$
\begin{aligned}
U \approx & m_t^{k,0}\left(\frac{\eta\varsigma_j\sigma_j}{2}\left((m_t^{j,l})_{|0}^{1+\sigma_j}\right)+\frac{1}{4}a_e^2\varsigma_j^2\frac{\sigma_j^2(1+\sigma_j)}{1+2\sigma_j}\left((m_t^{j,l})_{|0}^{1+2\sigma_j}\right)\right)\\
& - m_t^{k,0}\left(\frac{\eta\varsigma_j\sigma_j}{2}\left((m_t^{j,h})^{1+\sigma_j}\right)+\frac{1}{4}a_e^2\varsigma_j^2\frac{\sigma_j^2(1+\sigma_j)}{1+2\sigma_j}\left((m_t^{j,h})^{1+2\sigma_j}\right)\right)\\
& + m_t^{k,1}\left(\frac{\eta\varsigma_j\sigma_j}{2}\left((m_t^{j,h})^{1+\sigma_j}+(m_t^{j,m})_{|m_t^{k,0}}^{1+\sigma_j}\right)\right)\\
& + m_t^{k,1}\left(\frac{1}{4}a_e^2\varsigma_j^2\frac{\sigma_j^2(1+\sigma_j)}{1+2\sigma_j}\left((m_t^{j,h})^{1+2\sigma_j}+(m_t^{j,m})_{|m_t^{k,0}}^{1+2\sigma_j}\right)\right)\\
& + k_0\varsigma_k\sigma_k(m_t^{k,0})^{1+\sigma_k}+\frac{1}{2}b_e^2\varsigma_k^2\left(\frac{\sigma_k^2(1+\sigma_k)(m_t^{k,0})^{1+2\sigma_k}}{1+2\sigma_k}\right)\\
& + (1-m_t^{k,1})\left(\eta\varsigma_j\sigma_j(m_t^{j,h})^{1+\sigma_j}+\frac{1}{2}a_e^2\varsigma_j^2\left(\frac{\sigma_j^2(1+\sigma_j)(m_t^{j,h})^{1+2\sigma_j}}{1+2\sigma_j}\right)\right)\\
& + \log c_t(0,0)
\end{aligned}
$$

Additionally, if $\sigma_j = \sigma_k = 1$, and we set $m_t^{k,0} = 0$, which is the setup in the reduced model, the utility function simplifies in structure to:

$$
\begin{aligned}
U = & \int_0^{m_t^{j,m}}\int_0^{m_t^{k,1}} U_M d\psi d\phi + \int_0^{m_t^{j,h}}\int_{m_t^{k,1}}^{1} U_H d\psi d\phi\\
& + \int_{m_t^{j,m}}^{1}\int_0^{m_t^{k,1}} \log c_t(0,0) d\psi d\phi + \int_{m_t^{j,h}}^{1}\int_{m_t^{k,1}}^{1} \log c_t(0,0) d\psi d\phi
\end{aligned}
$$

In this case it is in fact possible to write out the utility as a polynomial in $m_t^{k,1}$:

$$
\begin{aligned}
U = & \log c_t(0,0) + \eta\varsigma_j(m_t^{j,h})^2 + \frac{1}{3}a_e^2\varsigma_j^2(m_t^{j,h})^3 \\
& + k_1\varsigma_k\left(\eta m_t^{j,h} + \frac{a_e^2\varsigma_j}{2}(m_t^{j,h})^2\right)(m_t^{k,1})^2 \\
& + \frac{1}{3}\left(k_1^2\varsigma_k^2\left(\frac{\eta}{\varsigma_j} + a_e^2 m_t^{j,h}\right) + b_e^2\varsigma_k^2\left(\eta m_t^{j,h} + \frac{a_e^2\varsigma_j}{2}(m_t^{j,h})^2\right)\right)(m_t^{k,1})^3 \\
& + \frac{1}{12}\left(6k_1 b_e^2\varsigma_k^3\left(\frac{\eta}{\varsigma_j} + a_e^2 m_t^{j,h}\right) + \frac{a_e^2\varsigma_k^3 k_1^3}{\varsigma_j}\right)(m_t^{k,1})^4 \\
& + \frac{1}{5}\left(b_e^4\varsigma_k^4\left(\frac{\eta}{\varsigma_j} + a_e^2 m_t^{j,h}\right) + \frac{a_e^2\varsigma_k^4 k_1^2 b_e^2}{\varsigma_j}\right)(m_t^{k,1})^5 \\
& + \frac{1}{6}\frac{a_e^2\varsigma_k^5 k_1 b_e^4}{\varsigma_j}(m_t^{k,1})^6 \\
& + \frac{4}{27}\frac{a_e^2\varsigma_k^8 b_e^6}{\varsigma_j}(m_t^{k,1})^9
\end{aligned}
$$

## B.3 Implementation in GiNaC

As previously, we use GiNaC to actually do the calculation, and store it for later use. The relevant section of the source code for the reduced model is:

```
ex utility()
{
    ex Utility =
    + (1 - mk1) * n * vsigmaj * (
        pow(mjh, 2)
    )
    + (1 - mk1) * pow(ae,2) * pow(vsigmaj,2) / 3 * (
        pow(mjh, 3)
    ) //the 1/2 cancels out with (1+sigmaj)=2.
    + n/(4 * vsigmaj) * intmjm2()
    + pow(ae,2)/(24 * vsigmaj) * intmjm3()
    + lc;

    return Utility;
}
```

# Appendix C

# The entries of $J_f$

The matrix $J_f$ in the reduced model represents the changes in employment levels as the marginals change. As mentioned in the text, the entries of the matrix are quite long, so they are reproduced on the next two pages in full generality. Due to the size of the entries, the pages are rendered in landscape to avoid having to cut up too many fractions.

$$
\begin{aligned}
J[1,1] =& 2\frac{\eta(\varsigma_k^3(m_t^{k,1})^3 b_e^2 + (m_t^{j,h})\varsigma_k(m_t^{k,1})\varsigma_j + \varsigma_k^2 k_1 (m_t^{k,1})^2) b_e^2}{\varsigma_j} \\
&+ 2\frac{(2\varsigma_k^4 k_1 (m_t^{k,1})^4 b_e^2 + 2(m_t^{j,h})\varsigma_k^3 (m_t^{k,1})^3 b_e^2 \varsigma_j + \varsigma_k^5 (m_t^{k,1})^5 b_e^4 + \varsigma_k^3 k_1^2 (m_t^{k,1})^3 + (m_t^{j,h})^2 \varsigma_k (m_t^{k,1})\varsigma_j^2 + 2(m_t^{j,h})\varsigma_k^2 k_1 (m_t^{k,1})^2 \varsigma_j) a_e^2 b_e^2}{\varsigma_j} \\
&+ \frac{a_e^2(\varsigma_k^2 k_1^2 (m_t^{k,1})^2 + 2\varsigma_k^3 k_1 (m_t^{k,1})^3 b_e^2 + 2(m_t^{j,h})\varsigma_k k_1 (m_t^{k,1})\varsigma_j + 2(m_t^{j,h})\varsigma_k^2 (m_t^{k,1})^2 b_e^2 \varsigma_j + (m_t^{j,h})^2 \varsigma_j^2 + \varsigma_k^4 (m_t^{k,1})^4 b_e^4)}{\varsigma_j} k_1 \\
&+ \frac{\eta((m_t^{j,h})\varsigma_j + \varsigma_k k_1 (m_t^{k,1}) + \varsigma_k^2 (m_t^{k,1})^2 b_e^2)}{\varsigma_j} k_1
\end{aligned}
$$

$$
\begin{aligned}
J[1,2] =& \frac{1}{3}\left(\frac{(3\varsigma_k k_1 (m_t^{k,1})^2 \varsigma_j + 2\varsigma_k^2 (m_t^{k,1})^3 b_e^2 \varsigma_j + 6(m_t^{j,h})(m_t^{k,1})\varsigma_j^2) a_e^2}{\varsigma_j} + 3\eta(m_t^{k,1})\right) k_1 \\
&+ \eta\varsigma_k (m_t^{k,1})^2 b_e^2 \\
&+ \frac{1}{3}\frac{a_e^2(3\varsigma_k^3 (m_t^{k,1})^4 b_e^2 \varsigma_j + 4\varsigma_k^2 k_1 (m_t^{k,1})^3 \varsigma_j + 6(m_t^{j,h})\varsigma_k (m_t^{k,1})^2 \varsigma_j^2) b_e^2}{\varsigma_j}
\end{aligned}
$$

$$
\begin{aligned}
J[2,1] =& -2\frac{\eta(\varsigma_k^3(m_t^{k,1})^3b_e^2 + (m_t^{j,h})\varsigma_k(m_t^{k,1})\varsigma_j + \varsigma_k^2k_1(m_t^{k,1})^2)b_e^2}{\varsigma_j} \\
& - (m_t^{j,h})^2a_e^2\varsigma_j - (m_t^{j,h})\eta \\
& + \frac{a_e^2(\varsigma_k^2k_1^2(m_t^{k,1})^2 + 2\varsigma_k^3k_1(m_t^{k,1})^3b_e^2 + 2(m_t^{j,h})\varsigma_kk_1(m_t^{k,1})\varsigma_j + 2(m_t^{j,h})\varsigma_k^2(m_t^{k,1})^2b_e^2\varsigma_j + (m_t^{j,h})^2\varsigma_j^2 + \varsigma_k^4(m_t^{k,1})^4b_e^4)}{\varsigma_j} \\
& - 2\frac{(2\varsigma_k^4k_1(m_t^{k,1})^4b_e^2 + 2(m_t^{j,h})\varsigma_k^3(m_t^{k,1})^3b_e^2\varsigma_j + \varsigma_k^5(m_t^{k,1})^5b_e^4 + \varsigma_k^3k_1^2(m_t^{k,1})^3 + (m_t^{j,h})^2\varsigma_k(m_t^{k,1})\varsigma_j^2 + 2(m_t^{j,h})\varsigma_k^2k_1(m_t^{k,1})^2\varsigma_j)a_e^2b_e^2}{\varsigma_j} \\
& + \frac{\eta((m_t^{j,h})\varsigma_j + \varsigma_kk_1(m_t^{k,1}) + \varsigma_k^2(m_t^{k,1})^2b_e^2)}{\varsigma_j} \\
& - \frac{a_e^2(\varsigma_k^2k_1^2(m_t^{k,1})^2 + 2\varsigma_k^3k_1(m_t^{k,1})^3b_e^2 + 2(m_t^{j,h})\varsigma_kk_1(m_t^{k,1})\varsigma_j + 2(m_t^{j,h})\varsigma_k^2(m_t^{k,1})^2b_e^2\varsigma_j + (m_t^{j,h})^2\varsigma_j^2 + \varsigma_k^4(m_t^{k,1})^4b_e^4)}{\varsigma_j}k_1 \\
& + \frac{\eta((m_t^{j,h})\varsigma_j + \varsigma_kk_1(m_t^{k,1}) + \varsigma_k^2(m_t^{k,1})^2b_e^2)}{\varsigma_j}k_1
\end{aligned}
$$

$$
\begin{aligned}
J[2,2] =& \frac{1}{3}\frac{(3\varsigma_kk_1(m_t^{k,1})^2\varsigma_j + 2\varsigma_k^2(m_t^{k,1})^3b_e^2\varsigma_j + 6(m_t^{j,h})(m_t^{k,1})\varsigma_j^2)a_e^2}{\varsigma_j} \\
& - \frac{1}{3}(\frac{(3\varsigma_kk_1(m_t^{k,1})^2\varsigma_j + 2\varsigma_k^2(m_t^{k,1})^3b_e^2\varsigma_j + 6(m_t^{j,h})(m_t^{k,1})\varsigma_j^2)a_e^2}{\varsigma_j} + 3\eta(m_t^{k,1}))k_1 \\
& - \eta\varsigma_k(m_t^{k,1})^2b_e^2 + \eta(m_t^{k,1}) - (\eta + 2(m_t^{j,h})a_e^2\varsigma_j)(-1 + (m_t^{k,1})) \\
& - \frac{1}{3}\frac{a_e^2(3\varsigma_k^3(m_t^{k,1})^4b_e^2\varsigma_j + 4\varsigma_k^2k_1(m_t^{k,1})^3\varsigma_j + 6(m_t^{j,h})\varsigma_k(m_t^{k,1})^2\varsigma_j^2)b_e^2}{\varsigma_j}
\end{aligned}
$$

# Appendix D

# The source code

The complete source code for GiNaC is provided below. Annotations are provided inline.

```
#include <iostream>
#include <ginac/ginac.h>
using namespace std;
using namespace GiNaC;

//Global variables
    possymbol psi("psi"),phi("phi");
    possymbol mk1("mk1", "(m_t^{k,1})");
    possymbol mjh("mjh" , "(m_t^{j,h})");
    realsymbol lc("lc", "\\log c_t(0,0)");
    possymbol vsigmak("vsigmak", "\\varsigma_k");
    possymbol vsigmaj("vsigmaj", "\\varsigma_j");
    possymbol n("n", "\\eta"),ae("ae", "a_e"),be("be", "b_e");
    realsymbol k1("k1", "k_1");
    realsymbol Fj("Fj", "F_j");
    realsymbol Fk("Fk", "F_k");


    possymbol H1("H1" ,"{H_t^1}") , H0("H0", "{H_t^0}");
    possymbol h1("h1" ,"{h_t^1}") , h0("h0", "{h_t^0}");
    possymbol C1("C1", "{C_t^1}") , C0("C0" ,"{C_t^0}");

    possymbol P("P" , "P_t");
    possymbol W1("W1" , "{W_t^1}") , W0("W0" , "{W_t^0}");

    possymbol kaK("kaK","\\kappa_k");
```

```
    possymbol kaJ("kaJ","\\kappa_j");
    possymbol kaK1("kaK1","\\kappa_{k1}");
    possymbol kaK0("kaK0","\\kappa_{k0}");
    possymbol kaJ1("kaJ1","\\kappa_{j1}");
    possymbol kaJ0("kaJ0","\\kappa_{j0}");

    possymbol muK("muK","\\mu_k");
    possymbol muJ("muJ","\\mu_j");
    possymbol muK1("muK1","\\mu_{k1}");
    possymbol muK0("muK0","\\mu_{k0}");
    possymbol muJ1("muJ1","\\mu_{j1}");
    possymbol muJ0("muJ0","\\mu_{j0}");



//SECTION OMITTED HERE, INCLUDED IN APPENDIX A



//Beware of division by zero if H1v is actually zero.
lst consumption(lst Employment)
{
   ex H1v = Employment[0];
   ex H0v = Employment[1];

   ex c11 = exp(
      Fj + vsigmaj * 2 * mjh
      + lc + Fk + vsigmak * 2 * mk1
   );
   ex c10 = exp( Fj + vsigmaj * 2 * mjh + lc );
   ex u0v = 1 - H1v - H0v;

   ex C1v = c11;

   ex C0v = ( H0v / (H0v + u0v) ) * c10
         + ( u0v / (H0v + u0v) ) * exp(lc);

   return lst( C1v, C0v );
}

matrix matr_H(lst Point, lst Employment, lst Testparams)
```

```
{

//Note we can use both the log
//and the plain employment levels below.
//Potentially useful.
   ex H1t = Employment[0].subs(Testparams);
   ex H0t = Employment[1].subs(Testparams);

   matrix J(2,2);
   J =
      H1t.diff(mk1).subs(Point), H1t.diff(mjh).subs(Point),
      H0t.diff(mk1).subs(Point), H0t.diff(mjh).subs(Point);


   return J;
}

matrix matr_h(lst Point, lst Employment, lst Testparams)
{

   ex h1t = log(Employment[0].subs(Testparams));
   ex h0t = log(Employment[1].subs(Testparams));

   matrix J(2,2);
   J =
      h1t.diff(mk1).subs(Point), h1t.diff(mjh).subs(Point),
      h0t.diff(mk1).subs(Point), h0t.diff(mjh).subs(Point);

   return J;
}


//Based on the general inverse jacobian,
//we produce a taylor-expansion in a
//specific point for the marginals.
//We also require the expression for the employments
//so that we can evaluate in these.
//We also produce the associated expression for log c_t(0,0).

lst marginals_H(matrix InvMatr_H, lst Point, lst Employment)
{
```

```
    ex mk1v =
        Point[0].rhs()
        + InvMatr_H[0]
            * (H1 - Employment[0].subs(Point))
        + InvMatr_H[1]
            * (H0 - Employment[1].subs(Point));
//Note the workaround to adress entries in a matrix.
//This is done in a serial fashion:
// -----
// |0,1|
// |2,3|
// -----
    ex mjhv =
        Point[1].rhs()
        + InvMatr_H[2]
            * (H1 - Employment[0].subs(Point))
        + InvMatr_H[3]
            * (H0 - Employment[1].subs(Point));

    ex Lcfunc = log(C1) - Fk - Fj
        - 2 * vsigmak * mk1v
        - 2 * vsigmaj * mjhv;

    return lst(
        mk1 == mk1v,
        mjh == mjhv,
        lc == Lcfunc
    );
}

lst marginals_h(matrix InvMatr_h, lst Point, lst Employment)
{
    ex mk1v =
        Point[0].rhs()
        + InvMatr_h[0]
            * (h1 - log(Employment[0].subs(Point)))
        + InvMatr_h[1]
            * (h0 - log(Employment[1].subs(Point)));

    ex mjhv =
        Point[1].rhs()
```

```
        + InvMatr_h[2]
            * (h1 - log(Employment[0].subs(Point)))
        + InvMatr_h[3]
            * (h0 - log(Employment[1].subs(Point)));

    ex Lcfunc = log(C1) - Fk - Fj
        - 2 * vsigmak * mk1v
        - 2 * vsigmaj * mjhv;

    return lst(
        mk1 == mk1v,
        mjh == mjhv,
        lc == Lcfunc
    );
}


//requires marginals exressed as functions of ht1, ht0
lst H0opt()
{

    ex H0opt = pow(
        (
        P * C1 * exp( - Fk  - 2 * vsigmak * muK )
        * pow(H1, -2 * vsigmak * muK1)
        * pow(H0, -2 * vsigmak * muK0)
        * pow(W0,-1)
        ) /
        (
        W0 * pow(H1, 2 * (vsigmak * muK1 + vsigmaj * muJ1) )
        )
    , 1 + 2 * (vsigmak * muK0 + vsigmaj * muJ0)) ;

    return lst( H0 == H0opt);
}


ex utility()
{

    ex Utility =
```

```
    + (1 - mk1) * n * vsigmaj * (
        pow(mjh, 2)
    )
    + (1 - mk1) * pow(ae,2) * pow(vsigmaj,2) / 3 * (
        pow(mjh, 3)
    ) //the 1/2 cancels out with (1+sigmaj)=2.
    + n/(4 * vsigmaj) * intmjm2()
    + pow(ae,2)/(24 * vsigmaj) * intmjm3()
    + lc;

    return Utility;

}


ex utilityS()
{

    ex Utility =
    lc
    + n * vsigmaj * pow(mjh,2)
    + pow(3,-1) * pow(ae * vsigmaj,2) * pow(mjh,3)
    + pow(3,-1) * pow(mk1,3) *
    (
    pow(k1 * vsigmak,2) *
        (
        n / vsigmaj + pow(ae,2) * mjh
        )
    + pow(be * vsigmak,2) *
        (
        n * mjh
        + 1/2 * vsigmaj * pow(ae * mjh,2)
        )
    );

    return Utility;

}


//////////////////////
```

```
//Checking functions//
//////////////////////

int sanity_check(lst Point, lst Testparams)
{

   cout << "max q for j=1: "
      << q1(0).subs(Point).subs(Testparams) << endl;
   cout << "max p for high: "
      << ph(0,mk1).subs(Point).subs(Testparams) << endl;
   cout << "max p for med: "
      << pm(0,0).subs(Point).subs(Testparams) << endl;
   cout << "implied mjm_0: "
      << mjm(0).subs(Testparams).subs(Point) << endl << endl;
   return 0;
}


int employment_check(lst Employment, lst Point, lst Testparams)
{

   cout << "H1 ="
      << Employment[0].subs(Point).subs(Testparams) << endl;
   cout << "H0 ="
      << Employment[1].subs(Point).subs(Testparams) << endl<< endl;

   return 0;
}

int consumption_check(lst Consumption, lst Point, lst Testparams)
{

   cout << "C1 ="
      << Consumption[0].subs(Point).subs(Testparams) << endl;
   cout << "C0 ="
      << Consumption[1].subs(Point).subs(Testparams) << endl<< endl;

   return 0;
}

int marginals_check(lst Marginals, lst Point2, lst Testparams)
```

```
{
   cout << "This means that the marginals
       as a function of employment levels are:" << endl;
   cout << "m_t^{k,1} =& "
      << Marginals[0].rhs().subs(Testparams) << endl;
   cout << "m_t^{j,h} =& "
      << Marginals[1].rhs().subs(Testparams) << endl;
   cout << "log c_t(0,0) =& "
      << Marginals[2].rhs().subs(Testparams) << endl;

   cout << "And with Point2 inserted:" << endl;
   cout << "m_t^{k,1} =& "
      << Marginals[0].rhs().subs(Point2)
          .subs(Testparams).evalf() << endl;
   cout << "m_t^{j,h} =& "
      << Marginals[1].rhs().subs(Point2)
          .subs(Testparams).evalf() << endl;
   cout << "log c_t(0,0) =& "
      << Marginals[2].rhs().subs(Point2)
          .subs(Testparams).evalf() << endl;

   return 0;
}

int main()
{
   /* Specify output type */
   cout << dflt;
// cout << latex;
// cout << csrc;


   /* Configurable */
   // The Point variable is the point
   // the Jacobian is inverted in to obtain
   // internal variables as functions
   // of the external variables.

   lst Point = lst(
      mk1 == 731*pow(10,-3),
```

```
        mjh == 731*pow(10,-3),
        lc == 0
    );

    // The implied external variables
    // based on the chosen internal point.
    lst Point2 = lst(
        H1 == employment()[0].subs(Point),
        H0 == employment()[1].subs(Point),
        C1 == consumption(employment())[0].subs(Point)
        );
// lst Point2 = lst();

    lst Point3 = lst(
        H1 == 0.20,
        H0 == 0.50,
        C1 == 14000
        );

    //    Note that ae^2+n<1 for the model to make sense.


    lst Testparams = lst(
    ae == 0.265,
    n == 0.43,

    Fj == 1.39,
    vsigmaj == 4.64,

    be == 0.2,
// k1 == 0.4,


    Fk == 3.8,
    vsigmak == 2

    );

// lst Testparams = lst();

// lst Testparams = lst(sigmak == 1);
```

```
// lst Testparams = lst(vsigmaj == 1, vsigmak == 1, sigmak == 1);


    /* Begin programme */

    sanity_check(Point,Testparams);

    lst Employment = employment();
    employment_check(Employment,Point,Testparams);

    lst Consumption = consumption(employment());
    consumption_check(Consumption,Point,Testparams);

    matrix InvMatr_H =
        (matr_H(Point, Employment, Testparams)).inverse();

    matrix InvMatr_h =
        (matr_h(Point, Employment, Testparams)).inverse();

    lst Marginals_H = marginals_H(InvMatr_H, Point, Employment);
    lst Marginals_h = marginals_h(InvMatr_h, Point, Employment);

    marginals_check(Marginals_H, Point2, Testparams);


//The below loop generates
// the table given in the calibration chapter.
for ( int i = 5; i >= 0; i--)
{

    lst Tp = lst(
    ae == 0.265,
    n == 0.43,

    Fj == 1.39,
    vsigmaj == 4.64,

    be == 0.2,
    k1 == 0.1 * i,
```

```
    Fk == 3.8,
    vsigmak == 2

    );

    lst Employment = employment();

    lst Consumption = consumption(employment());
    matrix InvMatr_H = (matr_H(Point, Employment, Tp)).inverse();
    matrix InvMatr_h = (matr_h(Point, Employment, Tp)).inverse();

    lst Marginals_H = marginals_H(InvMatr_H, Point, Employment);
    lst Marginals_h = marginals_h(InvMatr_h, Point, Employment);

    cout << i << endl;
    cout << Marginals_h[0].subs(Tp) << endl;
    cout << Marginals_h[1].subs(Tp) << endl;
}

    return 0;
}
```

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